Volumes of Tubes around Submanifolds of S^m

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This is a working document containing my study notes and examples.

1 Introduction

Let M and N be Riemannian manifolds and consider an isometric embedding $f : M^n \to P^m$. We are interested in the volume of the tube $T(N, r)$ of radius r around $M : T(M, r) = \{x \in P : d_P(x, M) \leq r\}$.

The volume can be computed using the normal bundle of N, as well as the second fundamental form and Jacobi fields. More precisely: let NM be the normal bundle of M inside P. Fix some point $p \in M$ and some normal vector $u \in NM$. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of S_u (the shape operator) and let $v_1, ..., v_n \in T_pM$ be the corresponding eigenvectors; we can choose the v_i's to be orthonormal. Choose $m - n - 1$ additional vectors $u_{n+1}, ..., u_{m-1}$ such that $v_1, ..., v_n, u_{n+1}, ..., u_{m-1}, u$ is an orthonormal basis of T_pP.

Consider the geodesic $\gamma(t) := exp_p(tu)$ and define Jacobi fields $J_1, ..., J_{m-1}$ along γ as follows:

$J_i(0) = v_i, J_i^{'}(0) = \lambda_i v_i$ for $i = 1..n$.

$J_i(0) = 0, J_i^{'}(0) = u_i$ for $i = n + 1, ..., m - 1$.

Finally, let $A(u, t)$ be the $(m - 1) \times (m - 1)$ matrix having as columns the vectors $J_1(t), ..., J_{m-1}(t)$, expressed in the basis $v_1, ..., v_n, u_{n+1}, ..., u_{m-1}$. Then:

$$\operatorname{Vol} T(M, r) = \int_{NM} \int_0^r detA(u, t)dtdVol$$

2 Submanifolds of $\mathbf{R}^m$

Jacobi fields in R^m are linear: $J(t) = J(0) + t \cdot J^{'}(0)$. In our particular case, $J_i(t) = (1 + \lambda_i t)v_i$ for $i = 1..n$ and $J_i(t) = tu_i$ for $i = n + 1, ..., m - 1$. Therefore, $detA(u, t) = t^{m-n-1}(1 + \lambda_1 t)...(1 + \lambda_n t)$, and the volume of the tube becomes:

$$\mathrm{Vol}\, T(M,r) = \int_{NM}\int_0^r t^{m-n-1}(1+\lambda_1 t)...(1+\lambda_n t)dtdVol$$

Example: $S^1 \subset \mathbf{R}^3$
to be typed

3 Submanifolds of S^m

We also have a nice formula for Jacobi fields on the sphere:
$J_i(t) = (cos(t) + \lambda_i sin(t))v_i$ for $i = 1..n$
$J_i(t) = sin(t)u_i$ for $i = n+1..m-1$.
This gives $detA(u,t) = \Pi_{i=1}^n (cos(t) + \lambda_i sin(t))(sin(t))^{m-n-1}$ and the volume of the tube:

$$\mathrm{Vol}\, T(M,r) = \int_{NM}\int_0^r \Pi_{i=1}^n (cos(t) + \lambda_i sin(t))(sin(t))^{m-n-1} dtdVol$$

Example: $S^1 \subset S^2$
to be typed

Example: $M^2 \subset S^3$
$detA(u,t) = (cos(t) + \lambda_1 sin(t))(cos(t) + \lambda_2 sin(t))$.

$$\int_0^r detA(u,t)dt = \int_0^r (cos(t) + \lambda_1 sin(t))(cos(t) + \lambda_2 sin(t))dt =$$

$$= \int_0^r cos^2(t) + (\lambda_1+\lambda_2)sin(t)cos(t) + \lambda_1\lambda_2 sin^2(t)dt =$$

$$= \frac{1}{2}t + \frac{1}{4}sin(2t) + \frac{1}{2}(\lambda_1+\lambda_2)sin^2(t) + \lambda_1\lambda_2(\frac{1}{2}t - \frac{1}{4}sin(2t))|_0^r$$

If $r = \pi$ then $\mathrm{Vol}\, T(M,\pi) = \int_{NM}\int_0^\pi detA(u,t)dtdVol =$

$$= \int_{NM} \frac{\pi}{2}(1+\lambda_1\lambda_2)dVol = \frac{\pi}{2}\int_{NM}(1+\lambda_1\lambda_2)dVol$$

But by Gauss' formula we have $1+\lambda_1\lambda_2 = K(p)$, the (intrinsic) Gaussian curvature of M at p, which does not depend on the unit normal vector (because $dimM = 2$). Since the codimension of this embedding is 1, there are exactly two unit normal vectors at every point $p \in M$, and the measure on NM is simply twice the measure on M. Therefore:

$\mathrm{Vol}\, T(M,\pi) = \frac{\pi}{2}\int_{NM}(1+\lambda_1\lambda_2)dVol = \frac{\pi}{2}\cdot 2\int_M K(p)dVol =$
$= \pi \cdot 2\pi\chi(M) = 2\pi^2\chi(M)$ by the Gauss-Bonnet theorem.

Conjecture. Let M be a submanifold of S^m. Then $\operatorname{Vol} T(M,\pi) = a_n\chi(M)$, where a_n is some factor depending on the dimension of M (perhaps something related to the volume of S_n?)

Example: $M^2 \subset S^4$ **- perhaps we need more terms**
$$\int_0^r sin(t)cos^2(t) + (\lambda_1+\lambda_2)sin^2(t)cos(t) + \lambda_1\lambda_2 sin^3(t)dt =$$
$$= -\tfrac{1}{3}cos^3(t)|_0^r + \tfrac{1}{3}(\lambda_1+\lambda_2)sin^3(t)|_0^r - \tfrac{1}{3}\lambda_1\lambda_2(2+sin^2(t)cos(t)|_0^r$$
For $r=\pi$ we obtain: $-\frac{1}{3}(-1-1)+0-\frac{1}{3}\lambda_1\lambda_2(-2-2) = \frac{4}{3}\lambda_1\lambda_2 + \frac{2}{3}$
Then $\operatorname{Vol} T(M,\pi) = \displaystyle\int_{NM} \frac{4}{3}\lambda_1\lambda_2 + \frac{2}{3}dVol =$
$= \frac{2}{3}\operatorname{Vol}(NM) + \frac{4}{3}\int_{NM}\lambda_1\lambda_2 dVol =$
$= \frac{2}{3}2\pi\operatorname{Vol}(M) + \frac{4}{3}\int_{NM}\lambda_1\lambda_2 dVol$

Using the extrinsic Gauss-Bonnet theorem for submanifolds of S^n (in preparation), we can express the integrand as a multiple of the Euler characteristic. I do not know how to interpret $1+\lambda_1\lambda_2$ in this case, since it depends on the unit normal vector u.

How does this fit in with the conjecture? Should there be an extra term in the conjecture? See the discussion about even-dimensional manifolds below.

Example: $M^3 \subset S^4$
$$\int_0^r \prod_{i=1}^{3}(cos(t)+\lambda_i sin(t)) =$$
$$= \int_0^r cos^3(t) + cos^2(t)sin(t)(\lambda_1+\lambda_2+\lambda_3) +$$
$$+cos(t)sin^2(t)(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3) + \lambda_1\lambda_2\lambda_3 sin^3(t)dt =$$
$$= \tfrac{1}{3}(2+cos^2(t))sin(t)|_0^r - \tfrac{1}{3}(\lambda_1+\lambda_2+\lambda_3)cos^3(t)|_0^r +$$
$$+(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3)\tfrac{1}{3}sin^3(t)|_0^r - \tfrac{1}{3}\lambda_1\lambda_2\lambda_3(2+sin^2(t))cos(t)|_0^r$$
With $r=\pi$ we obtain:
$$-\tfrac{1}{3}(\lambda_1+\lambda_2+\lambda_3)(-1-1) - \tfrac{1}{3}\lambda_1\lambda_2\lambda_3(-2-2) =$$
$$= \tfrac{2}{3}(\lambda_1+\lambda_2+\lambda_3) + \tfrac{4}{3}\lambda_1\lambda_2\lambda_3 =$$
$= \frac{2}{3}tr(S_u) + \frac{4}{3}det(S_u)$ where S_u is the shape operator.

This doesn't look too promising, until we realize that $dimM = 3$ implies $det(S_u) = -det(S_{-u})$ and $tr(S_u) = -tr(S_{-u})$. Integrating over NM we obtain $\operatorname{Vol} T(M,\pi) = 0 = \chi(M)$, since M is a compact, 3-dimensional manifold.

Idea: $M^n \subset S^m$ **for odd** n

By Poincare duality we know that $\chi(M) = 0$ for odd-dimensional, closed manifolds.

Let P_k be the k-th degree symmetric polynomial in the λ_i's. It seems that $\mathrm{Vol}\, T(M\pi) = \sum_{k=0}^{n} T_k \int_{NM} P_k dVol$ for some quantities T_k which depend on the integration of various sines and cosines. But $\int_{NM} P_k dVol = 0$ for all odd k's, so it suffices to show that $T_k = 0$ for all even k's. This should be just a calculus argument.

Theorem. If M^n is an odd-dimensional, compact manifold without boundary immersed in S^m then $\mathrm{Vol}\, T(M, \pi) = 0$.

Proof. Let $P_k(x_1, ..., x_l)$ denote the k-degree symmetric polynomial in the variables $x_1, ..., x_l$. Then:

$$detA(u,t) = \prod_{i=1}^{n} sin^{m-n-1}(t)(cos(t) + \lambda_i sin(t)) =$$
$$= \sum_{k=0}^{n} P_k(\lambda_1, ..., \lambda_n) \cdot cos^{n-k}(t) sin^{m+k-n-1}(t)$$

$$\mathrm{Vol}\, T(M, \pi) = \int_{NM} \int_0^{\pi} detA(u,t) dt dVol =$$
$$= \sum_{k=0}^{n} \int_{NM} P_k(\lambda_1, ..., \lambda_n) \int_0^{\pi} cos^{n-k}(t) sin^{m+k-n-1}(t) dt dVol =$$
$$= \sum_{k=0}^{n} F(n,k) \int_{NM} P_k(\lambda_1, ..., \lambda_n) dVol$$

where $F(n,k) = \int_0^{\pi} cos^{n-k}(t) sin^{m+k-n-1}(t) dt$

For n odd and k even, $n-k$ is odd and we can integrate:

$$F(n,k) = \int_0^{\pi} cos^{n-k}(t) sin^{m+k-n-1}(t) dt =$$
$$= \int_0^{\pi} (cos^2(t))^{\frac{n-k-1}{2}} sin^{m+k-n-1}(t) cos(t) dt =$$
$$= \int_0^{\pi} (1 - sin^2(t))^{\frac{n-k-1}{2}} sin^{m+k-n-1}(t) cos(t) dt =$$

$= \int_0^{\pi} Q(sin(t)) cos(t) dt$ where Q is some polynomial. But $\int Q(t) dt$ is a polynomial with no term of degree 0, so $F(n,k)$ will be a polynomial in $sin(t)$ with no free term, evaluated from 0 to π. Since $sin(0) = sin(\pi) = 0$, we can conclude that $F(n,k) = 0$ for n odd and k even.

On the other hand, when k is odd, we have $\int_{NM} P_k(\lambda_1, ..., \lambda_n) dVol = 0$, so all the terms of $T(M, \pi)$ are zero. □

The case $M^n \subset S^m$ for even n

$$\mathrm{Vol}\,T(M,\pi) = \sum_{k=0}^{n} \int_{NM} P_k(\lambda_1, ..., \lambda_n) dVol \cdot \int_0^\pi cos^{n-k}(t) sin^{m+k-n-1}(t)dt$$

If k is odd then $n-k$ is even, so $\int_0^\pi cos^{n-k}(t)sin^{m+k-n-1}(t)dt = 0$. See the discussion for n odd.

If k is even: assume (by embedding $S^m \subset S^{m+1}$ if necessary) that m is even. Then $m+k-n-1$ is odd.

Computation:

$$\int_0^\pi cos^{2a}(t)sin^{2b+1}(t)dt = \int_0^\pi cos^{2a}(t)(1-cos^2(t))^b sin(t)dt =$$

$$= -\int_1^{-1} u^{2a}(1-u^2)^b du = \int_{-1}^{1} u^{2a} \sum_{i=0}^{b} \binom{b}{i} (-1)^i u^{2i} du =$$

$$= \sum_{i=0}^{b} \binom{b}{i} (-1)^i \int_{-1}^{1} u^{2(a+i)} du = \sum_{i=0}^{b} \binom{b}{i} (-1)^i \frac{u^{2(a+i)+1}}{2(a+i)+1} =$$

$= \sum_{i=0}^{b} \binom{b}{i} \frac{(-1)^i}{a+i+1/2}$

Lemma. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{x+k} = \frac{n!}{a(a+1)...(a+n}$

Proof. Partial fractions. □

Applying this lemma, we obtain:

$$\int_0^\pi cos^{2a}(t)sin^{2b+1}(t)dt = \frac{b!}{(a+1/2)(a+3/2)...(a+n+1/2)}$$

In the problem at hand, we have $a = \frac{n-k}{2}$ and $b = \frac{m+k-n-2}{2}$, which give:

$$\int_0^\pi cos^{n-k}(t)sin^{m+k-n-1}(t)dt = \frac{(\frac{m+k-n-2}{2})!}{\frac{n-k+1}{2} \cdot \frac{n-k+3}{2} \cdots \frac{m-1}{2}} > 0$$

The moral of this story / computation is that we actually encounter all the even-degree basic symmetric polynomials in the λ_i's. The questions is - how do these integrate over NM?

4 Stuff to think about

Do these formulas work for manifolds with boundary? Recall Weyl's definition of a tube (basically ignore the boundary completely).

Is the conjecture (about the Euler characteristic) just a result of formulas, or does it have some other meaning? What is the interpretation of $\mathrm{Vol}\,T(M,\pi) = 0$ in some of the results? How is volume counted / interpreted?

What is the proof of the formula for the volume of a tube, in general? (review)

5 References