| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | **Total** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | |
| | | | | | | | | | | | | |

D o n o t w r i t e a b o v e t h i s l i n e !

**YOUR NAME** (print):

**MATH 114 – 001/002**

**Make-up 1st midterm**

November 19, 2012

**Your Section/Instructor's Name** (circle one): 001/Pop 004/Cooper

**Your TA** (first name):

**Rules**:

- A single $8\frac{1}{2}$ by 11 inch handwritten page (one sided) is permitted.
- No other written or printed materials or electronic devices are allowed.

**Grading**:

- There are 8 problems (with suggested answers) and 4 questions each worth 10 points.
- Do all problems, **showing your work**, and *circling* your answers.
- **This is not a multiple choice exam!** No credit will be given if you circle the right answer, but do not show the work leading to the answer.

**Instructions**:

- Be prepared to show your Penn ID if asked to do so.
- Write you name at the top of each page of the exam.
- *Do not detach* any of the pages of the exam.

**Academic Integrity Statement:**

*My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this Math 114 Midterm Exam.*

Name (printed): Solutions Signature:

NAME:

1. Consider the vector $\vec{v} = 2\vec{\imath} - \vec{\jmath}$, and the lines: $L_1$ through the point $(0,1)$ and parallel to $\vec{v}$, respectively the line $L_2$ through the point $(2,10)$ and perpendicular to $\vec{v}$. The intersection point of $L_1$ and $L_2$ is:

(a) $(0,3)$ (b) $(2,0)$ (c) $(-1,1)$ (d) $(0,0)$ (e) $(-2,2)$ (f) $(1,1)$

$\vec{w} = \vec{\imath} + 2\vec{\jmath}$ is $\perp$ to $\vec{v}$.

$L_1$: $\vec{w} \cdot \langle x-0, y-1 \rangle = 0$

$x + 2(y-1) = 0$

$L_2$: $\vec{v} \cdot \langle x-2, y-10 \rangle = 0$

$2(x-2) - (y-10) = 0$

$x + 2y - 2 = 0$

$2x - y + 6 = 0$

$5x + 10 = 0$

$x = -2$

$-2 + 2y - 2 = 0$

$2y = 4$

$y = 2$

2. Find $\vec{r}(1)$, provided $\vec{r}(t)$ satisfies:

$$\frac{d^2\vec{r}}{dt^2} = (-t^2, 1, -t), \qquad \frac{d\vec{r}}{dt}(1) = \left(\tfrac{2}{3}, 0, -\tfrac{1}{2}\right), \qquad \vec{r}(0) = (1, -1, 0).$$

a) $(\frac{23}{12}, -\frac{3}{2}, \frac{1}{3})$ b) $(2, -\frac{1}{2}, 0)$ c) $(2, 1, 0)$ d) $(2, -1, 1)$ e) $(2, 0, -1)$ f) $(\frac{23}{2}, 0, 0)$

$\frac{d\vec{r}}{dt} = \langle -\frac{1}{3}t^3 + a, t + b, -\frac{1}{2}t^2 + c \rangle = \langle -\frac{1}{3}t^3 + 1, t - 1, -\frac{1}{2}t^2 + \frac{1}{2} \rangle$

$\frac{d\vec{r}}{dt}(1) = \langle -\frac{1}{3} + a, 1 + b, -\frac{1}{2} + c \rangle = \langle \frac{2}{3}, 0, 0 \rangle$

$a = 1, \; b = -1, \; c = \frac{1}{2}$

$\vec{r}(t) = \langle -\frac{1}{12}t^4 + t + d, \frac{1}{2}t^2 - t + e, -\frac{1}{6}t^3 + \frac{1}{2}t + f \rangle$

$\vec{r}(0) = \langle d, e, f \rangle = \langle 1, -1, 0 \rangle$

$\vec{r}(t) = \langle -\frac{1}{12}t^4 + t + 1, \frac{1}{2}t^2 - t - 1, -\frac{1}{6}t^3 + \frac{1}{2}t \rangle$

$\vec{r}(1) = \langle 2 - \frac{1}{12}, \frac{1}{2} - 2, -\frac{1}{6} + \frac{1}{2} \rangle$

NAME:

3. The point on the intersection line of the planes $2x - y - z = 2$ and $x + y + z - 1 = 0$ which is closest to the point $(9, 0, -1)$ is:

(a) $(9, 1, \frac{1}{2})$ (b) $(0, 9, -\frac{1}{2})$ (c) $(-1, -\frac{1}{2}, 0)$ (d) $(1, \frac{1}{2}, -\frac{1}{2})$ (e) $(-1, -\frac{1}{2}, -\frac{3}{2})$ (f) $(2, 0, 9)$

$\vec{n}_1 = \langle 2, -1, -1 \rangle \quad \vec{n}_2 = \langle 1, 1, 1 \rangle$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = i \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} - j \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + k \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$= \langle 0, -3, 3 \rangle$ direction of line of intersection

Point on intersection

$(1, 0, 0)$

$2x - y - z = 2$

$x + y + z = 1$

$3x = 3$

$t = \frac{-6}{2 \cdot 18} = \frac{-1}{6}$

$L: \langle 1, -3t, 3t \rangle$

$d^2(t) = 8^2 + (-3t)^2 + (-1-3t)^2 = 65 + 18t^2 + 6t$

4. Which of the following planes contains the $x$-axis and is perpendicular on the plane $3x - y + z = 20$ ?

(a) $y - 2z = 0$ (b) $2y + z = 0$ (c) $x = -z$ (d) $z = 1$ (e) $x + 2y = 0$ (f) $2y = -2z$

Contains x-axis: $(x, 0, 0)$ solves for any $x$.

a), b), f)

$\vec{n} = \langle 3, -1, 1 \rangle$

$\vec{n}_a = \langle 0, 1, -2 \rangle$

$\vec{n}_b = \langle 0, 2, 1 \rangle$

$\vec{n}_f = \langle 0, 2, 2 \rangle$

$\vec{n} \cdot \vec{n}_f = 0$

NAME:

5. Find the distance from the $x$-axis to the intersection line of the planes $x+y+z-1=0$ and $-2x-2y+z=1$.

(a) $-1$ (b) $0$ (c) $1$ (d) $\sqrt{2}$ (e) $\sqrt{3}$ (f) $\sqrt{4}$

$\vec{n}_1 = \langle 1,1,1\rangle \quad \vec{n}_2 = \langle -2,-2,1\rangle$

$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -2 & -2 & 1 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ -2 & 1\end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ -2 & 1\end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ -2 & -2\end{vmatrix}$

$= 3i - 3j = \langle 3,-3,0\rangle$

$x+y+z-1=0$

$-2x-2y+z-1=0$

$3z-3=0$

$z=1$

$L: \langle 3t,-3t,1\rangle$

distance$^2$ to x-axis:

$d^2(t) = (y(t))^2 + (z(t))^2$

$= 9t^2+1$ smallest value is 1 (when $t=0$.)

6. Let $L$ be the line tangent to the trajectory $\vec{r}(t) = \left(\cos^2(t), \sin(t)+1, -\frac{1}{2}t^2+t+2\right)$ at the point $\vec{r}(0)$. Then the angle (in radians) of $L$ with the $xy$-plane is:

(a) $0$ (b) $\pi/2$ (c) $\pi/3$ (d) $\pi/4$ (e) $\pi/6$ (f) $-\pi/2$

$\vec{r}\,'(t) = \langle -2\sin t\cos t, \cos t, -t+1\rangle$

$\vec{r}\,'(0) = \langle 0,1,1\rangle \qquad \vec{n} = \langle 0,0,1\rangle$

angle between $\vec{r}\,'(0)$ and $\vec{n}$ : $\arccos\left(\frac{1}{\sqrt{2}\cdot 1}\right) = \frac{\pi}{4}$

angle between $\vec{r}\,'(0)$ and xyplane $\frac{\pi}{2} - \pi/4 = \pi/4$

NAME:

7. Consider the unit vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ having all the components positive numbers, and satisfying: $\vec{u}_1$ is parallel to the line through the origin and $(-3,-1,-1)$; $\vec{u}_2$ is parallel the intersection line of $x-y-3z=0$ and $-x+2y+2z=0$, and $\vec{u}_3$ is a multiple of $3\vec{\imath}+\vec{\jmath}+2\vec{k}$. Then the volume of the parallelepiped spanned by $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is:

a) $\frac{1}{4\sqrt{63}}$ b) $\frac{1}{5\sqrt{33}}$ c) $\frac{1}{\sqrt{331}}$ d) $\frac{1}{10\sqrt{21}}$ e) $\frac{1}{6\sqrt{77}}$ f) $\frac{1}{12\sqrt{13}}$

Let $v_1 = \langle 3,1,1\rangle$, $v_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & -3 \\ -1 & 2 & 2 \end{vmatrix} = i\begin{vmatrix} -1 & -3 \\ 2 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$

$v_3 = \langle 3,1,2\rangle$. $= \langle 4,1,1\rangle$

Then $U_1 = \frac{v_1}{|v_1|}$ $U_2 = \frac{v_2}{|v_2|}$ $U_3 = \frac{v_3}{|v_3|}$

$(U_1 \times U_2)\cdot U_3 = \frac{1}{|v_1||v_2||v_3|}(v_1 \times v_2)\cdot v_3$

$= \frac{1}{\sqrt{11}\sqrt{18}\sqrt{14}}\begin{vmatrix} 3 & 1 & 2 \\ 4 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = \frac{1}{6\sqrt{77}}\cdot\left(3\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix}\right)$

$= \frac{1}{6\sqrt{77}}\cdot(0-1+2)$

8. Consider the statements:
   (i) The curvature of the curve $x=3\cos(t), y=5+3\sin(t), z=1$ is equal to $\frac{1}{2}$.
   (ii) If the acceleration of a motion $\vec{r}(t) = (x(t), y(t), z(t))$ is everywhere zero, then the trajectory of the motion is a circle.

Which of the following assertions is true?

a) (i) only b) (ii) only c) (i) and (ii) d) neither of (i) and (ii) e) (ii) is true if $t<0$ f) none of the above

i) Circle of radius $3 \Rightarrow K = \frac{1}{3}$

ii) $\vec{r}(t) = \langle t,t,t\rangle$ has $\vec{r}\,'(t) = \langle 1,1,1\rangle$ $\vec{r}\,''(t) = \vec{0}$

But $\vec{r}$ traces a line.

NAME:

9. Find the value of the $x$-coordinate where the plane through the points $(4,1,1)$, $(1,2,1)$, and $(1,1,2)$ intersects the $x$-axis.

a) 14 b) 10 c) 8 d) 5 e) 3 f) 1

$\vec{v} = \langle 3, -1, 0 \rangle \quad \vec{w} = \langle 3, 0, -1 \rangle$

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 3 & -1 & 0 \\ 3 & 0 & -1 \end{vmatrix} = i \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - j \begin{vmatrix} 3 & 0 \\ 3 & -1 \end{vmatrix} + k \begin{vmatrix} 3 & -1 \\ 3 & 0 \end{vmatrix}$$

$$= i + 3j + 3k$$

$P: \; 1(x-1) + 3(y-1) + 3(z-2) = 0$ x-axis: $y = z = 0$

$$x - 1 + 3(-1) + 3(-2) = 0$$

$$x = 1 + 3 + 6 = 10$$

10. The space curves define by the following vector-valued functions $\vec{r}(t) = \left(t, \sin(t), t^4\right)$ and $\vec{s}(t) = \left(t^3, t, \sin(t)\right)$ intersect at the point $\vec{r}(0) = (0,0,0) = \vec{s}(0)$. Then the angle (in radians) between the two curves at the point $(0,0,0)$ is:

a) 0 b) $\pi/6$ c) $\pi/4$ d) $\pi/3$ e) $\pi/2$ f) $2\pi/3$

$\vec{r}\,'(t) = \langle 1, \cos t, 4t^3 \rangle \quad \vec{r}\,'(0) = \langle 1, 1, 0 \rangle$

$\vec{s}\,'(t) = \langle 3t^2, 1, \cos t \rangle \quad s'(0) = \langle 0, 1, 1 \rangle$

$\vec{s}\,'(0) \cdot \vec{r}\,'(0) = 1 = \sqrt{2}\sqrt{2} \cos\theta$

So $\theta = \arccos \frac{1}{2} = \pi/3$

NAME:

11. Answer whether the following is true or false, and give a **reason / counterexample**:

The vector $(\vec{\jmath} \times (\vec{k} \times \vec{\imath})) \times \vec{\imath}$ is a unit vector, i.e., has length one.

$$(\vec{j} \times (\vec{k} \times \vec{i})) \times \vec{i} = (\vec{j} \times (\vec{j})) \times \vec{i} = 0 \times \vec{i} = \vec{0}$$

FALSE

12. Answer whether the following is true or false, and give a **reason / counterexample**:

(a) If $\vec{v} \perp \vec{w}$, then $3\vec{v} + 2\vec{w}$ and $-3\vec{v} + 2\vec{w}$ have the same length. TRUE

(b) If $3\vec{v} + 2\vec{w}$ and $-3\vec{v} + 2\vec{w}$ have the same length, then $\vec{v} \perp \vec{w}$. TRUE

$$|3\vec{v}+2\vec{w}|^2 = (3\vec{v}+2\vec{w})\cdot(3\vec{v}+2\vec{w})$$

$$= 9\vec{v}\cdot\vec{v} + 6\vec{v}\cdot\vec{w} + 6\vec{w}\cdot\vec{v} + 4\vec{w}\cdot\vec{w}$$

$$= 9|\vec{v}|^2 + 12\vec{v}\cdot\vec{w} + 4|\vec{w}|^2$$

$$|-3\vec{v}+2\vec{w}|^2 = 9|\vec{v}|^2 - 12\vec{v}\cdot\vec{w} + 4|\vec{w}|^2,$$

a) If $\vec{v} \perp \vec{w}$, then $\vec{v}\cdot\vec{w}$, so both lengths are $9|\vec{v}|^2 + 4|\vec{w}|^2$

b) If lengths are equal, $12\vec{v}\cdot\vec{w} = -12\vec{v}\cdot\vec{w}$

So $\vec{v}\cdot\vec{w} = 0$, hence $\vec{v} \perp \vec{w}$