1	2	3	4	4	5	6	7	8	9	10	11	12	**Total**

D o n o t w r i t e a b o v e t h i s l i n e !

YOUR NAME (print):

MATH 114 – 001/002

Midterm # 2

November 13, 2012

Your Section/Instructor's Name (circle one): 001/Pop 002/Cooper

Your TA (first name):

Rules:

- A single $8\frac{1}{2}$ by 11 inch handwritten page (one sided) is permitted.
- No other written or printed materials or electronic devices are allowed.

Grading:

- There are 8 problems (with suggested answers) and 4 questions each worth 10 points.
- Do all problems, **showing your work**, and *circling* your answers.
- **This is not a multiple choice exam!** No credit will be given if you circle the right answer, but do not show the work leading to the answer.

Instructions:

- Be prepared to show your Penn ID if asked to do so.
- Write you name at the top of each page of the exam.
- *Do not detach* any of the pages of the exam.

Academic Integrity Statement:

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this Math 114 Midterm Exam.

Name (printed): Solutions Signature:

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1. Let L be the line tangent to the ellipse $2x^2 + y^2 = 3$ at the point $(1, 1)$. Then the point on L which is closest to $(2, 4)$ is:

(a) $(0, 3)$ (b) $(-1, 2)$ (c) $(-1, 0)$ (d) $(0, 0)$ (e) $(-1, -1)$ (f) $(1, 1)$

def $f(x,y) = 2x^2 + y^2$. $\nabla f = \langle 4x, 2y \rangle$ $\nabla f|_{1,1} = \langle 4, 2 \rangle$

L: $4(x-1) + 2(y-1) = 0$ $y = -2x + 3$

$d^2(x,y) = (x-2)^2 + (-2x+3-4)^2$

$= x^2 - 4x + 4 + 4x^2 + 4x + 1 = 5x^2 + 5$

Smallest when $x = 0$: $(0, 3)$

2. Let $f(x, y) = x^2 \sin(2y)$, and let $\mathcal{P}$ be the plane tangent to the curve $z = f(x, y)$ at the point defined by $(x, y) = (1, \frac{\pi}{4})$. Which of the following planes is parallel to $\mathcal{P}$?

(a) $-2x - y = 1$ (b) $x + y + 2z = 1$ (c) $-2x + z = 1$

(d) $-x + 2y = 1$ (e) $2x + y + z = 1$ (f) $-2x + y + z = 1$

$\nabla f = \langle 2x \sin 2y, 2x^2 \cos 2y \rangle$

$\nabla f|_{1,\pi/4} = \langle 2 \sin \frac{\pi}{2}, 2 \cos \pi/2 \rangle = \langle 2, 0 \rangle$

$\mathcal{P}$: $2(x-1) + 0(y - \frac{\pi}{4}) - (z - f(1, \frac{\pi}{4})) = 0$

normal: $\langle 2, 0, -1 \rangle$

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3. Let $\ell(x,y) = x\ln(y^2 + \frac{3}{4})$. Then $\ell(x,y)$ achieves a local maximum at:

(a) $(0,\frac{1}{2})$ (b) $(0,-\frac{1}{2})$ (c) $(1,0)$ (d) $(1,1)$ (e) $(2,1)$ (f) nowhere

$\nabla \ell = \langle \ln(y^2+\frac{3}{4}), \frac{2xy}{y^2+\frac{3}{4}} \rangle$

cp: $\ln(y^2+\frac{3}{4}) = 0$

$y^2 + \frac{3}{4} = 1$

$y = \frac{1}{2}$,

$x = 0$.

$$D^2\ell = \begin{pmatrix} 0 & \frac{2y}{y^2+\frac{3}{4}} \\ \frac{2y}{y^2+3/4} & \frac{4y^2 - 2(y^2+\frac{3}{4})}{(y^2+\frac{3}{4})^2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{2y}{y^2+\frac{3}{4}} \\ \frac{2y}{y^2+\frac{3}{4}} & \frac{2y^2-3/4}{y^2+3/4} \end{pmatrix}$$

$D = \frac{-4y^2}{(y^2+\frac{3}{4})^2} < 0$ when $y = \frac{1}{2}$

So $(0,\frac{1}{2})$ is a saddle point.

4. One of the tangent planes to the surface $x^2 + 2xy + y^2 + 2x - z + 2 = 0$ which contains the x-axis is:

(a) $y + z = 0$ (b) $2y + z = 0$ (c) $x + z = 0$ (d) $z = 1$ (e) $x + 2y = 0$ (f) $2y = 1$

[**Hint:** Which planes $ax + by + cz = d$ contain the x-axis?...]

x-axis: $\{(x,0,0)\}$ so a), b) contain the x-axis

$f(x,y,z) = x^2 + 2xy + y^2 + 2x - z + 2$

$\nabla f = \langle 2x+2y+2, 2x+2y, -1 \rangle$

Must have $2x+2y+2 = 0$, so $2x+2y = -2$, so

$\langle 0, -2, -1 \rangle$ is normal to the tangent plane

when the tangent plane contains the x-axis.

$2y + z = c$

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5. The distance from the point $(0, 1, 2)$ to the surface $x^2 + y^2 - 4z = 0$ is:

(a) -1 (b) 0 (c) 1 (d) $\sqrt{2}$ (e) $\sqrt{3}$ (f) 2

$d^2(x,y,z) = x^2+(y-1)^2+(z-2)^2$ $\quad g(x,y) = x^2+y^2-4z$

$\nabla d^2 = \langle 2x, 2(y-1), 2(z-2)\rangle$ $\quad \nabla g = \langle 2x, 2y, -4\rangle$

$\nabla d^2 = \lambda \nabla g$

$2x = \lambda 2x$ — If $x \neq 0$, $\lambda = 1$. So $2(y-1) = 2y$ no sol'n. So $x = 0$.

$2(y-1) = 2\lambda y$ — If $y=0$, $-2 = 0 \rightarrow\!\!\!\!\times$ so $y \neq 0$, so

$2z-4 = -4\lambda$

$\lambda = 1 - \frac{1}{y} = 1 - \frac{z}{2}$ $\quad \frac{1}{y} = \frac{z}{2}$ $\quad y = \frac{2}{z}$

$\left(\frac{2}{z}\right)^2 - 4z = 0$ $\quad \frac{4}{z^2} = 4z$ $\quad 1 = z^3$ $\quad z = 1, y = 2, x = 0$

$d^2(0,2,1) = 0+1+1 = 2$

6. The sum of the absolute maximum and the absolute minimum of the function $g(x, y) = x^2 + 2xy^2 - 2x$ on the region $\{(x, y) \mid x^2 + y^2 \leq 2\}$ is:

(a) $1 - \sqrt{3}$ (b) $1 + \sqrt{3}$ (c) $1 - \sqrt{8}$ (d) $1 + \sqrt{8}$ (e) $1 + \sqrt{13}$ (f) $1 + \sqrt{13}$

$\nabla g = \langle 2x+2y^2-2, 4xy\rangle$

$\nabla g = 0$ when $2x+2y^2-2 = 0$

$4xy = 0$

If $x=0$, $2y^2 = 2$ $\quad y = \pm 1$

If $y = 0$, $2x = 2$ $\quad x = 1.$

c.p. $(0,1), (0,-1), (1,0)$

all interior.

$g(0,1) = g(0,-1) = 0$

$g(1,0) = -1$

For boundary, see next page

boundary: $h(x,y) = x^2+y^2-2=0$

$\nabla g = \lambda \nabla h$

$2x+2y^2-2 = 2\lambda x$

$4xy = 2\lambda y$

$y=0 \Rightarrow x^2=2 \Rightarrow x = \pm\sqrt{2}$

$y \neq 0 \Rightarrow 2x = \lambda$

$x+y^2-2=2x^2$

$x+(2-x^2)-2=2x^2$

$3x^2-x-1=0$

$x = \frac{1\pm\sqrt{13}}{6}$

$g(\pm\sqrt{2},0)$
$=2\pm2\sqrt{2}$
$\approx 4.8,$
$-.8$

$0<\frac{1+\sqrt{13}}{6} < \frac{5}{6}$, so

$g(x) = x^2+2x(2-x^2)+2x$
$= -2x^3+x^2+2x$

has $g\left(\frac{1+\sqrt{13}}{6}\right) < \left(\frac{5}{6}\right)^2 + 2\cdot\frac{5}{6}$
$< 3 < 4.8$

$g\left(\frac{1+\sqrt{13}}{6}\right) > 0$ since when $|x|<1$,
$|x^3| < |x^2| < |x|$.

$-\frac{3}{6} < \frac{1-\sqrt{13}}{6} < 0$

$g\left(\frac{1-\sqrt{13}}{6}\right) < 2\cdot\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^2 = \frac{2}{8}+\frac{1}{4} = \frac{1}{2} < 1$

$g\left(\frac{1-\sqrt{13}}{6}\right) > 2\cdot -\frac{1}{2} = -1$

So absolute max is $2+2\sqrt{2}$, absolute min is -1

$2+2\sqrt{2}-1 = 1+2\sqrt{2} = 1+\sqrt{8}$

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7. The temperature of a metal plate in the xy-coordinates is given by $T(x, y) = x\sin(2y)$. Suppose the probe is moving along the circle of radius 1 centered at the origin, according to $x = \cos(2t), y = \sin(2t)$. How fast is the temperature probe's reading changing when it reaches the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$?

a) $\cos(\sqrt{3})$ b) $\sin(2\sqrt{3})$ c) $\cos(\sqrt{3}) - \sqrt{3}\sin(\sqrt{3})$ d) 0 e) $\frac{1}{2}\sin(\sqrt{3})$ f) $\cos(2) + \frac{1}{2}\cos(\frac{\sqrt{3}}{2})$

$$\frac{d}{dt}\left(T(x(t), y(t))\right) = \nabla T \cdot \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$$

$$= \langle \sin 2y, 2x\cos 2y \rangle \cdot \langle -2\sin 2t, 2\cos 2t \rangle$$

$$= \langle \sin 2y, 2x\cos 2y \rangle \cdot \langle -2y, 2x \rangle$$

$$= -2y\sin 2y + 4x^2\cos 2y \; : \quad \text{when } x = \tfrac{1}{2}\ y = \tfrac{\sqrt{3}}{2},$$

$$\frac{d}{dt}\left(T(x(t), y(t))\right) = -\sqrt{3}\sin\sqrt{3} + \cos\sqrt{3}$$

8. Consider all the rectangular boxes of sides a, b, c satisfying $a^2 + 4b^2 + 9c^2 \le 108$. The maximal possible volume of such boxes is:

(a) 25 (b) 27 (c) 29 (d) 33 (e) 36 (f) 39

$V = abc \qquad \nabla V = \langle bc, ac, ab \rangle = 0$

when $bc = 0$, $ac = 0$, $ab = 0$

$\Rightarrow$ at least one of $a, b, c = 0$. So interior local extrem value is 0, probably not a maximum. Check boundary!

$$\langle bc, ac, ab \rangle = \lambda \langle 2a, 8b, 18c \rangle$$

$bc = 2\lambda a \qquad ac = 8\lambda b \qquad$ So $3a^2 = 108$

$ac = 8\lambda b \qquad ab = 18\lambda c \qquad a^2 = 36$

$\frac{b}{a} = \frac{1}{4}\frac{a}{b} \qquad \frac{c}{b} = \frac{8}{18}\frac{b}{c} \qquad a = 6$

$4b^2 = a^2 \qquad 9c^2 = 4b^2 \qquad b = 3$

$c = 2$

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9. The altitude of a hill is described by the function $f(x,y) = \frac{1}{2}x^2y - \pi x\cos(y)$, where x is how far east the point is and y is how far north the point is. Is it true that a ball released from rest at the point on the hill corresponding to $(1,\pi)$ will start rolling due north?

- **Justify your answer!**

Ball rolls downhill, i.e. in the direction $-\nabla f$.

$-\nabla f|_{1,\pi} = -\langle xy - \pi\cos y, \frac{1}{2}x^2 + \pi x\sin y\rangle|_{1,\pi}$

$= -\langle \pi - \pi(-1), \frac{1}{2} + 0\rangle = \langle -2\pi, -\frac{1}{2}\rangle$

Not due north! (west-southwest)

10. Let a function $h(x,y)$ have a saddle point at (x_0,y_0). Is it true that the Hessian determinant $h_{xx}(x_0,y_0)\,h_{yy}(x_0,y_0) - h_{xy}(x_0,y_0)^2$ of $h(x,y)$ at (x_0,y_0) must be negative?

- **Justify your answer!**

No. E.g. $h_1(x,y) = x^4 - y^4$ or

$h_2(x,y) = |x| - |y|$ at $(x,y) = (0,0)$.

(h_2 is not diff'ble, and

$(h_1)_{xx} = 12x^2 \quad (h_1)_{yy} = -12y^2$

$(h_1)_{xy} = 0$ So at $(0,0)$, Hess. det. $= 0$.)

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11. Find out whether the limit

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\sin(xy)}{x^4+y^4}$$

exits, and if the limit exists, compute the limit.

Approach along positive x-axis:

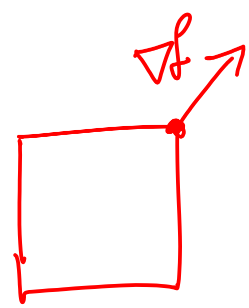
$$\lim_{x\to 0^+} \frac{x^2 \sin(0)}{x^4} = 0$$

Approach along $y=x$ from 1st quadrant:

$$\lim_{x\to 0^+} \frac{2x^2 \sin(x^2)}{2x^4} = \lim_{x\to 0} \frac{\sin x^2}{x^2} = 1.$$

Limit does not exist.

12. Is it true that the point $(1,1)$ is a local maximum of the function



$$f(x,y) = e^{\sin^2(xy)} + \ln(1+x^4+y^4)$$

in the region $-1 \le x, y \le 1$, but inspite of that one has that $f_x(a,b) \neq 0$?

- **Justify your answer!**

Yes. $\nabla f = \langle 2y\sin(xy)\cos(xy)e^{\sin^2(xy)} + \frac{4x^3}{1+x^4+y^4},$

$2x\sin(xy)\cos(xy)e^{\sin^2(xy)} + \frac{4y^3}{1+x^4+y^4} \rangle$

At $(x,y)=(1,1)$, ∇f points in the same dir. as $\langle 1,1\rangle$, so $f_x>0$ and $f_y>0$ near $(1,1)$. Since every point (x,y) in the square has $x\le 1$, $y\le 1$, this means $f(1,1) \ge f(x,y)$. for (x,y) near $(1,1)$.