Math 114 MC Midterm 1 Exam Solutions

Problem 1. Suppose $y(t)$ satisfies the logistical equation

$$\frac{dy}{dt} = 2y - ky^2$$

where k is a constant. Suppose you know that for the initial condition $y(0) = \frac{1}{10}$ the solution $y(t) \to 3$ as $t \to \infty$. What is the value of k?

Solution: If you are very familiar with these problems, you might want to write $2y - ky^2 = y(2 - ky)$. Then, you see that $\frac{dy}{dt} = 0$ whenever $y = 0$ or $y = \frac{2}{k}$. In other words, the slope is 0 whenever the function is equal to 0 or to $\frac{2}{k}$. If you remember what the graphs looked like in section 9.4, you could deduce that if the slope is 0 at $y = 0$ or $y = \frac{2}{k}$, then 0 and $\frac{2}{k}$ are asymptotes of $y(t)$. But by the statement we know that 3 is an asymptote and $\frac{2}{k}$ can never be equal to 0 so it must be equal to 3. Hence, $\frac{2}{k} = 3$ so $k = \frac{2}{3}$.

If you are not as familiar with this problem, then you might want to solve it a more standard way: make phase lines for the derivative and the second derivative like you did on the homework (I will leave that part to you). Once you get the phase lines, you can sketch the curve. Again you will have 0 and $\frac{2}{k}$ are asymptotes and in fact you will be able to see from the graph that the function is increasing when we have the initial condition $y(0) = \frac{1}{10}$. Not only will $y(t)$ be increasing, but it will be increasing to the asymptote that is Not 0, namely it will be increasing towards $\frac{2}{k}$ as $t \to \infty$. So again we get the same answer of $k = \frac{2}{3}$.

Problem 2. Find the $x-$coordinate of the point on the plane

$$3x + 2y + z = 13$$

closest to the point $(x, y, z) = (1, 1, 1)$.

Solution: Here you should know how to find the distance from a point to the plane. Not for the actual solution, but to understand what we are measuring when we measure the distance of the closest point. When we measure distance of the closest point, we are taking the distance of The orthogonal vector from the plane to that point. One way to think about this is if you are living on the plane and you see the point above you, then the closest distance from you to that point is a line going straight above your head to that point.

Once you understand this, the goal becomes to find the vector from the plane to your point. This will give us the point on the plane closest to $(1, 1, 1)$. The only method I know how to do this is to see that $(3, 2, 1)$ is parallel to all other orthogonal vectors to our plane. Therefore, we need to solve $(3, 2, 1) \times (a-1, b-1, c-1) = 0$. (a, b, c) will then be the point on our plane that we are looking for, and in fact we only need to solve for a to get the answer (it is asking for the $x-$coordinate). Let's compute:

$$\begin{aligned} 0 = (3,2,1) \times (a-1, b-1, c-1) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ a-1 & b-1 & c-1 \end{vmatrix} \\ = (2(c-1) - (b-1), (a-1) &- 3(c-1), 3(b-1) - 2(a-1)). \end{aligned}$$

So we have:

$$\begin{aligned} &2(c-1) = b-1; \\ &a-1 = 3(c-1); \\ &3(b-1) = 2(a-1). \end{aligned}$$

We can use these equation and the fact that $3a + 2b + c = 13$ to solve for a. From the system of equations we see $c = \frac{a+2}{3}$ and $b = \frac{2a+1}{3}$. If we try to solve the system using only those three equations, we will fail to do so. We need to plug them back into the equation of the plane to get

$$3a + 2\frac{2a+1}{3} + \frac{a+2}{3} = 13 \implies 14a = 35 \implies a = \frac{5}{2}.$$

This is the answer we are looking for.

Problem 3. Find the plane consisting of all points equidistant to the points $(1,1,1)$ and $(3,-3,5)$ and determine where this plane intersects the $x-$axis.

There are two ways I can think of doing this problem. I will start with the long way.

Solution 1: Some students love the midpoint formula. Unfortunately, it is hard to think of a problem for which the midpoint formula will be the most useful technique. This technique is not the best for this problem. If this way makes the most sense to you, maybe you would like your solution to look like the following:

It is very easy to see that the midpoint must be on the plane (we are looking for a plane equidistant from both points). Hence, $(2,-1,3)$ must lie on our plane. Now all we need is a normal vector and we can find the equation of the plane. It takes a little intuition to realize that a normal vector to our plane would go from $(2,-1,3)$ to either one of $(1,1,1)$ or $(3,-3,5)$. Hence, we can take our normal vector to be $\mathbf{n} = (2-1,-1-1,3-1) = (1,-2,2)$. Now we compute

$$0 = \mathbf{n} \cdot (x-2, y+1, z-3) = (x-2) - 2(y+1) + 2(z-3) = x - 2y + 2z - 10.$$

Hence, our plane is

$$x - 2y + 2z = 10.$$

We are trying to solve for when $(a,0,0)$ is on our plane; a will be the $x-$intersection. So plug this into our plane to get $a = 10$ is the $x-$intercept.

It is important to note that we could have plugged in $y = 0, z = 0$ at any time and simply solved for x.

Solution 2: This one is much faster. The problem is talking about distance and whenever distance is mentioned you should be reminded of the distance formula. Namely, you could do

$$\sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2} = \sqrt{(x-3)^2 + (y+3)^2 + (z-5)^2}$$

and solve for the equation of the plane, then plug in $(a,0,0)$ where a will be the answer to the problem. *However, again there is a much simpler way: plug in $y = z = 0$ into the equation above and solve for x.* We get:

$$\begin{aligned}
&\sqrt{(x-1)^2 + 1 + 1} = \sqrt{(x-3)^2 + 9 + 25} \\
&\implies (x-1)^2 = (x-3)^2 + 32 \\
&\implies -2x + 1 = -6x + 9 + 32 \\
&\implies 4x = 40 \implies x = 10.
\end{aligned}$$

So $x = 10$ is the answer.

Problem 4. Find the $x-$coordinate of the point on the line through the points $(1, 1, 1)$ and $(3, 2, 0)$ that is closest to the origin.

Solution: You might want to try a couple of things for this problem. We know how to get the distance from any point to our line. However, this will not work. We need to use the distance formula again.

First, get an equation for our line: $\ell(t) = (1, 1, 1) + t(2, 1, -1) = (2t + 1, t + 1, -t + 1)$. Now the distance from $\ell(t)$ to the origin is given by

$$d(t) = \sqrt{(2t + 1)^2 + (t + 1)^2 + (-t + 1)^2} = \sqrt{6t^2 + 4t + 3}.$$

We are looking for the minimum distance, so you want to find $d'(t)$ and set it equal to 0. Note that we have a function of only t instead of the usual x, y and z, so this method will be effective.

$$d'(t) = \frac{6t + 2}{d(t)}.$$

$$0 = d'(t) \implies 6t + 2 = 0 \implies t = \frac{-1}{3}.$$

Plugging in $t = \frac{-1}{3}$ into our $x-$component, we get $x = \frac{1}{3}$.

Problem 5. Find the point where the plane containing the points $(1,1,1),(2,3,2)$ and $(3,2,4)$ intersects the $x-$axis.

Solution: This problem is pretty standard. We want to find the equation of the plane containing the three points mentioned above, then plug in $y = z = 0$ to get the $x-$coordinate.

$$(3,2,4) - (1,1,1) = (2,1,3)$$
$$(3,2,4) - (2,3,2) = (1,-1,2).$$
$$\text{Next, } (2,1,3) \times (1,-1,2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 1 & -1 & 2 \end{vmatrix} = (5,-1,-3).$$

Therefore the equation of our plane is

$$0 = (5,-1,-3) \cdot (x-1, y-1, z-1) = 0.$$

To make life simpler, we can again just plug in $y = z = 0$ right away and solve for x :

$$5(x-1) + 1 + 3 = 0 \implies x = \frac{1}{5}.$$

Problem 6. A particle is moving along the path given by

$$\mathbf{r}(t) = (1+t, 2-2t, 2+2t).$$

How fast is the particle moving away from the origin at time $t = 0$? I.e. what is $\frac{d}{dt}||\mathbf{r}(t)||$ at $t = 0$?

Solution: Same story: use the distance formula. The hint already gives you what to do. We need to find how fast it is moving away, i.e. the rate of change of distance. We compute:

$$\frac{d}{dt}||\mathbf{r}(t)|| = \frac{d}{dt}\sqrt{(1+t)^2 + (2-2t)^2 + (2+2t)^2} = \frac{d}{dt}\sqrt{9t^2 + 2t + 9} = \frac{9t+1}{\sqrt{9t^2 + 2t + 9}}.$$

Plugging in $t = 0$ we see that the rate of change of distance is $\frac{1}{3}$.

Problem 7. What is the curvature of the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{16} = 1$$

at $x = 2, y = 0$? Hint: the hyperbola can be parametrized by $x(t) = e^t + e^{-t}, y(t) = 2e^t - 2e^{-t}$. At $t = 0$ we get $x = 2, y = 0$.

Solution: It is a good idea to follow the hint. After using the hint, we can use the following formula for curvature:

$$\kappa(t) = \frac{|\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)|}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}.$$

We can calculate:

$$\dot{x}(t) = e^t - e^{-t}; \qquad \ddot{x}(t) = e^t + e^{-t}.$$
$$\dot{y}(t) = 2e^t + 2e^{-t}; \qquad \ddot{y}(t) = 2e^t - 2e^{-t}.$$

Therefore

$$\dot{x}(0) = 0; \qquad \ddot{x}(0) = 2.$$
$$\dot{y}(0) = 4; \qquad \ddot{y}(t) = 0.$$

Plugging into the formula, we get

$$\kappa(0) = \frac{8}{4^3} = \frac{1}{8}.$$

Solution 2: We can compute the curvature using only the formula

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{\frac{3}{2}}}.$$

We can solve for x as a function of y near $x = 2$: $x(y) = (\sqrt{16 + y^2})/2$. It doesn't matter whether we have a function of x or y; curvature at a point does not depend on the parametrization, only the shape of the curve near that point. Here, we get $x'(0) = 0$ and $x''(0) = \frac{1}{8}$. Using the formula, we again get $\kappa(x = 2, y = 0) = \frac{1}{8}$.

Problem 8. Find the arc length of the curve

$$\mathbf{r}(t) = (t, t^2, \frac{2t^3}{3})$$

from $t = 0$ to $t = 3$.

Solution: Arc length problems tend to be some of the easiest. We just need to know the definition and use it:

$$\int_0^3 \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2} dt = \int_0^3 \sqrt{1 + 4t^2 + 4t^4} dt = \int_0^3 (2t^2 + 1) dt = \frac{2t^3}{3} + t \Big|_0^3 = 21.$$

Problem 9. Assume the acceleration of gravity is $10\frac{m}{s^2}$ downward. A cannon is fired at ground level with initial velocity $v(t) = (30, 40)\frac{m}{s}$. What is the maximum curvature of the path of the cannonball. Hint: maximum curvature occurs at the highest point.

Solution: We are given that $a(t) = (0, -10)$. Hence,

$$v(t) = (c_1, c_2 - 10t) \text{ and } r(t) = (c_1 t + d_1, d_2 + c_2 t - 5t^2).$$

We are given $v(0) = (30, 40)$ so $c_1 = 30, c_2 = 40$. Moreover, we may assume we are starting at the origin, so

$$r(0) = (0, 0) \implies d_1 = d_2 = 0.$$

Hence,

$$r(t) = (30t, 40t - 5t^2).$$

To find the highest point, you want to take the derivative of the $y-$component and set it equal to zero. You will get that $t = 4$. Now you can use the first formula for curvature given in the solution of Problem 7 to get $\kappa(t = 4) = \frac{1}{90}$.

Once again if you do not want to use this formula, we can explicitly solve for $r(t)$ as a function of x. We see that the function is

$$y = f(x) = \frac{240x - x^2}{180}.$$

Then

$$f'(x) = \frac{120 - x}{90}.$$

So $f'(x) = 0$ at $x = 120$. So we want to find the curvature of $f(x)$ at $x = 120$. Compute one more derivative to get $f''(120) = -\frac{1}{90}$. Moreover we have already seen $f'(120) = 0$. So

$$\kappa(x = 120) = \frac{|f''(120)|}{(1 + f'(120)^2)^{\frac{3}{2}}} = \frac{\frac{1}{90}}{1} = \frac{1}{90}.$$

Problem 10. Assume the acceleration of gravity is $10\frac{m}{s^2}$ downward. A watermelon cannon is fired at ground level and the watermelon rises to a height of 10 meters and travels a distance of 40 meters horizontally before it strikes the ground. What is the initial speed when the watermelon left the cannon?

Solution 1: This would be a pretty good algebraic solution, but there is a more intuitive one.

We have

$$r(t) = (v\cos(\alpha)t, v\sin(\alpha)t - 5t^2)$$

where α is the angle the watermelon was launched, and v is the initial speed. So we are trying to solve for v.

We also know that $y_{\max}$ occurs at $t = \frac{v\sin(\alpha)}{10}$. But $y_{\max} = 10$ by assumption. Hence,

$$10 = v\sin(\alpha) \cdot \frac{v\sin(\alpha)}{10} - 5 \cdot \frac{v^2\sin^2(\alpha)}{100} \implies v\sin(\alpha) = 10\sqrt{2}.$$

But the $x-$ coordinate of the maximum is the midpoint of $x = 0$ and $x = 40$ (we are given it hits the ground at $x = 40$). Hence, we also have

$$20 = v\cos(\alpha) \cdot \frac{v\sin(\alpha)}{10} = v\cos(\alpha) \cdot \sqrt{2} \implies v\cos(\alpha) = 10\sqrt{2}.$$

Then we get the solution:

$$v^2 = (v\cos(\alpha))^2 + (v\sin(\alpha))^2 = 200 + 200 = 400 \implies v = 20.$$

Solution 2: You can also try to think about the shape the function makes. You can actually see that it is a parabola with zeroes at $x = 0$ and $x = 40$. If we consider the parametrization, then the function is given by

$$f(x) = \tan(\alpha)x - \frac{5x^2}{(v\cos(\alpha))^2}.$$

But we also know that any parabola that represents the path a projectile makes with zeroes and $x = 0$ and $x = 40$ is given by $cx(40-x)$ for some constant c. Since the maximum is 10, and the maximum of $f(x)$ is at $x = 20$, we see that $c = \frac{1}{40}$. We can now take the derivative of $f(x) = \frac{x(40-x)}{40}$ and see that the slope is 1. Hence, $\tan(\alpha) = 1$ so we have $v\sin(\alpha) = v\cos(\alpha)$. Finally, we again use that the max occurs at $x = 20$ except now we use

$$0 = f'(20) = \tan(\alpha) - \frac{10 \cdot 20}{(v\cos(\alpha)^2)} \implies v^2\sin(\alpha)\cos(\alpha) = 200 \implies v\sin(\alpha) = v\cos(\alpha) = \sqrt{200}.$$

The last implication only holds because we already knew $v\sin(\alpha) = v\cos(\alpha)$. But again we see $v^2 = 200 + 200 \implies v = 20$.