- Find the $n = 0$ Bessel series for $f(x) = 1$ on $[0, 2]$ with the Neumann boundary condition.

  - The $n = 0$ Bessel basis is $y(x) = J_0(\alpha_i x)$ where $y'(2) = \alpha_i J_0'(2\alpha_i) = 0$.
  - $y_1(x) = 1$ solves the Bessel equation with $\alpha = 0$, and satisfies the boundary condition $y_1'(2) = 0$.
  - The norm-squared of $y_1 = 1$ is
  $$\int_0^2 1 \cdot 1 \cdot x dx = \frac{1}{2} 2^2 = 2$$
  - The inner product of $y = 1$ with f is:
  $$\int_0^2 1 \cdot 1 \cdot x dx = \frac{1}{2} 2^2 = 2$$
  - To find the other eigenvalues, we note that the recurrence relation
  $$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$
  applied with $n = 0$ says that $J_0' = -J_1$. So the α must be chosen such that
  $$-J_1(2\alpha) = 0$$
  i.e. $\alpha = \frac{1}{2} j_{1,i}$ for some i.
  - Thus the eigenvalues are $\{0, \frac{j_{1,1}}{2}, \frac{j_{1,2}}{2}, \ldots\}$.
  - Using the book's computation, the norm-squared of the eigenfunction $y(x) = J_0(\frac{j_{1,i}}{2} x)$ is:
  $$\|J_0(\frac{j_{1,i}}{2} x)\|^2 = \frac{2^2}{2} J_0^2(\frac{j_{1,i}}{2} 2) = 2 J_0^2(j_{1,i})$$
  - We also compute the inner product of f with y:
  $$\int_0^2 x J_0(\frac{j_{1,i}}{2} x) dx = \frac{4}{j_{1,i}^2} \int_0^{j_{1,i}} u J_0(u) du$$
  where $u = \frac{j_{1,i}}{2} x$. Using (with $n = 1$) the recurrence relation
  $$\frac{d}{du} [u^n J_n] = u^n J_{n-1}$$
  we can continue:
  $$\frac{4}{j_{1,i}^2} \int_0^{j_{1,i}} u J_0(u) du = \frac{4}{j_{1,i}^2} \int_0^{j_{1,i}} \frac{d}{du} [u J_1(u)] du = \frac{4}{j_{1,i}^2} [u J_1(u)|_{u=0}^{u=j_{1,i}} = 0$$
  since by definition $J_1(j_{1,i}) = 0$.
  - Thus the Bessel series is:
  $$1 \cdot 1 + \sum_{i=1}^{\infty} 0 J_0(\frac{j_{1,i}}{2} x)$$

  Note that we should have expected this, since the function $f(x) = 1$ is itself one of the Neumann Bessel eigenfunctions!

- Find the $n = 0$ Bessel series for $f(x) = 1$ on $[0, 2]$ with the Dirichlet boundary condition.

  - The $n = 0$ Bessel basis is $y(x) = J_0(\alpha_i x)$ where $y(2) = J_0(2\alpha_i) = 0$.
  - The eigenvalues are $\{\frac{j_{0,1}}{2}, \frac{j_{0,2}}{2}, \ldots\}$.
  - We use the computation in class to see that the coefficient of $J_0(\frac{j_{0,i}}{2}x)$ is

$$\begin{aligned} c_i &= \frac{2}{2^2 J_1^2(j_{0,i})} \int_0^2 x J_0(\frac{j_{0,i}}{2}x) dx \\ &= \frac{2}{2^2 J_1^2(j_{0,i})} \frac{4}{j_{0,i}^2} \int_0^{j_{0,i}} u J_0(u) du \\ &= \frac{2}{2^2 J_1^2(j_{0,i})} \frac{4}{j_{0,i}^2} [u J_1(u)|_{u=0}^{u=j_{0,i}} \\ &= \frac{2}{j_{0,i} J_1^2(j_{0,i})} J_1(j_{0,i}) \\ &= \frac{2}{j_{0,i} J_1(j_{0,i})} \end{aligned}$$

    where we used the same trick from above.
  - So the Bessel series is:

$$f(x) = \sum_{i=1}^{\infty} \frac{2}{j_{0,i} J_1(j_{0,i})} J_0(\frac{j_{0,i}}{2}x)$$