# Laplace's Equation
# textbook sections 13.2, 13.5

MATH 241

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## Laplace's Equation

We want to describe how the temperature $u$ in metal sheet at thermal equilibrium.

assumption the sheet is homogeneous

assumption the sheet is perfectly insulated (except at the edges)

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Laplace's equation is elliptic.

If we insulate the $x$-edges, then no heat can leave across them. So

$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0 \qquad\qquad \frac{\partial u}{\partial x}\Big|_{x=a} = 0$$

This is the Neumann boundary condition.

If we cool the $x$-edges, then

$$u(0, y) = 0 \qquad\qquad u(a, y) = 0$$

This is the Dirichlet boundary condition.

We also need to specify $u(x, b)$ and $u(x, 0)$

## Finding Separable Solutions of Laplace's Equation

1. separate variables to get $X'' + \lambda X = 0,\ Y'' - \lambda Y = 0.$
2. use boundary conditions to find $\lambda$ (SLP!)
3. recognise that

$$\sum_{n=0}^{\infty} \left(a_n \cos(\alpha_n x) + b_n \sin(\alpha_n x)\right)\left(c_n \cosh(\alpha_n y) + d_n \sinh(\alpha_n y)\right)$$

   formally solves Laplace's equation
4. Use the sine and cosine series for the boundary data $u(x, 0)$ and $u(x, b)$ to find the coefficients.

## Superposition Principle

The sum of two solutions of a linear equation is also a solution of the equation.

## Mixed Boundary Values

To solve Laplace's equation with $u(x,0) = f(x)$, $u(x,b) = g(x)$, $u(0,y) = F(y)$, $u(a,y) = G(y)$:

1. Solve Laplace's equation with $u(x,0) = f(x)$, $u(x,b) = g(x)$, $u(0,y) = 0$, $u(a,y) = 0$. Call this solution $u_1$.
2. Solve Laplace's equation with $u(x,0) = 0$, $u(x,b) = 0$, $u(0,y) = F(y)$, $u(a,y) = G(y)$. Call this solution $u_2$.
3. The solution we're looking for is $u = u_1 + u_2$.

## Note

$$u_1(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a}x\right) \left( C_n \cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right) \right)$$

$$u_2(x,y) = \sum_{n=1}^{\infty} \left( C_n \cosh\left(\frac{n\pi}{b}x\right) + D_n \sinh\left(\frac{n\pi}{b}x\right) \right) B_n \sin\left(\frac{n\pi}{b}y\right)$$

## Solving Laplace's Equation on Semi-infinite Domains

To solve Laplace's equation with $0 < x < a$, $y > 0$, note that

$$Y(y) = c_1 e^{-\alpha y} + c_2 e^{\alpha y}$$

solves $Y'' - \alpha^2 Y = 0$, but we need $c_2 = 0$ to ensure that $Y$ is bounded.

Solutions are of the form:

$$\sum_{n=0}^{\infty} (A_n \cos(\alpha_n x) + B_n \sin(\alpha_n x))\, C_n e^{-\alpha_n y}$$