The Wave Equation
textbook sections 13.2, 13.4

MATH 241

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The Wave Equation

We want to describe how the displacement u of a string varies over time.

assumption spatially-uniform tension

assumption tension acts tangentially

assumption gravity is negligible

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The wave equation is hypberbolic.

Generally we want to keep the ends of the string fixed.

$$u(0,t) = 0 \qquad\qquad u(L,t) = 0$$

This is the fixed boundary condition.

If we let the ends of the string slide along parallel tracks,

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad\qquad \frac{\partial u}{\partial x}(L,t) = 0$$

This is the free end condition.

We also need to specify:

initial displacement $u(x,0)$

initial velocity $\frac{\partial u}{\partial t}(x,0)$

Finding Separable Solutions of the Wave Equation

1. separate variables to get $X'' + \lambda X = 0,\ Y'' + \lambda Y = 0.$
2. use boundary conditions to find λ (SLP!)
3. recognise that

$$\sum_{n=0}^{\infty} (a_n \cos(\alpha_n x) + b_n \sin(\alpha_n x))\,(c_n \cos(\alpha_n t) + d_n \sin(\alpha_n t))$$

   formally solves the wave equation
4. Use the sine and cosine series for the initial data $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$ to find the coefficients.

d'Alembert's solution

1. Change variables to $\xi = x + t, \eta = x - t$.
2. We can write $u(x,t) = F(\xi) + G(\eta)$.
3. Integrate boundary conditions to get:

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds$$

Note: this works on any domain, but may cause trouble at the endpoints (cf. problem 8 of the practice exam).

Summary of Separable Solutions to Physical PDE

heat	Laplace	wave
$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$
initial data, boundary data	boundary data	initial data
$e^{-\alpha^2 t}\text{trig}(\alpha x)$	$\text{trig}(\alpha x)\text{trigh}(\alpha y)$	$\text{trig}(\alpha x)\text{trig}(\alpha t)$

in more variables

The Laplacian of $u(x_1, \ldots, x_n)$ is $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$.

n-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \Delta u$$

n-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

n-dimensional Laplace's equation:

$$\Delta u = 0$$