Polar and Cylindrical Coordinates
textbook sections 14.1, 14.2

MATH 241

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the disc

Assume the polar ansatz:

$$u(r,\theta) = R(r)\Theta(\theta)$$

Boundary data can be given by specifying $u(R_0,\theta)$ or $\frac{\partial u}{\partial r}(R_0,\theta)$.

For the disc, we need $\Theta(\theta) = \Theta(\theta + 2\pi)$.

polar Laplacian

The Laplacian in polar coordinates is given by:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Laplace's equation on the disc (Dirichlet)

1. separate variables to get $\Theta'' + \lambda\Theta = 0$, $r^2 R'' + rR' - \lambda R = 0$.
2. Θ must satisfy periodic boundary conditions with period 2π.
3. $\Theta(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta)$
4. recognise that

   $$A_0 + \sum_{n=1}^{\infty} (A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta))$$

   formally solves Laplace's equation
5. Use Fourier series for the boundary data $u(R_0, \theta)$ to find the coefficients A_n, B_n:

   $$A_0 = \frac{a_0}{2} \qquad A_n = R_0^{-n} a_n \qquad B_n = R_0^{-n} b_n$$

   where a_n and b_n are the Fourier coefficients for $u(R_0, \theta)$.

Laplace's equation on the disc (Neumann)

1. separate variables to get $\Theta'' + \lambda\Theta = 0$, $r^2 R'' + rR' - \lambda R = 0$.
2. Θ must satisfy periodic boundary conditions with period 2π.
3. $\Theta(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta)$
4. recognise that

$$A_0 + \sum_{n=1}^{\infty} \left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right)$$

formally solves Laplace's equation

5. Use the Fourier series for the boundary data $\frac{\partial u}{\partial r}(R_0, \theta)$ to find the coefficients A_n, B_n:

$$A_0 \text{ is free} \qquad A_n = \frac{a_n}{nR_0^{n-1}} \qquad B_n = \frac{b_n}{nR_0^{n-1}}$$

where a_n and b_n are the Fourier coefficients for $\frac{\partial u}{\partial r}(R_0, \theta)$.

Note

For the problem

$$\begin{cases} \Delta u & = 0 \\ \frac{\partial u}{\partial r}(R_0, \theta) & = g(\theta) \end{cases}$$

not all boundary data g are admissible!

Need $\int_0^{2\pi} g(\theta) d\theta = 0$.

Laplace's equation on the semidisc

1. separate variables to get $\Theta'' + \lambda\Theta = 0$, $r^2 R'' + rR' - \lambda R = 0$.
2. Use boundary conditions to get a SLP for Θ.
3. recognise that

$$A_0 + \sum_{n=1}^{\infty} \left(A_n r^{\alpha_n} \cos(\alpha_n \theta) + B_n r^{\alpha_n} \sin(\alpha_n \theta)\right)$$

   formally solves Laplace's equation
4. Use sine or cosine series for the boundary data to find coefficients.

radial symmetry

If we have a perfectly round drum, which we strike in the dead centre with a perfectly round mallet, we expect the subsequent vibrations to be radially symmetric.

We represent this mathematically by working in polar coordinates (r, θ), and assuming that

$$\frac{\partial u}{\partial \theta} = 0$$

radial 2d wave equation

If we assume the radial symmetry ansatz, the displacement $u(r,t)$ of the drumhead satisfies:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

The fixed-boundary condition is:

$$u(R_0, t) = 0$$

We also need to specify initial data:

initial displacement $u(r,0)$

initial velocity $\frac{\partial u}{\partial t}(r,0)$

Finding separable solutions to the radial 2d wave equation

1. Separation gives $T'' + \lambda T = 0$, $rR'' + R' + r\lambda R = 0$
2. With $R(R_0) = 0$, this gives $\lambda = \frac{j_{0,i}^2}{R_0^2}$
3. Recognise that

$$\sum_{i=1}^{\infty} \left[A_i \cos\left(\frac{j_{0,i}}{R_0} t\right) + B_i \sin\left(\frac{j_{0,i}}{R_0} t\right) \right] J_0\left(\frac{j_{0,i}}{R_0} r\right)$$

   formally solves the 2d wave equation.
4. Use the $n = 0$ Bessel series for the boundary data to find A_i and B_i:

$$A_i = a_i \qquad\qquad B_i = \frac{R_0 b_i}{j_{0,i}}$$

   where a_i is the $n = 0$ Dirichlet Bessel coefficient for $u(r, 0)$ and b_i is the $n = 0$ Dirichlet Bessel coefficient for $\frac{\partial u}{\partial t}(r, 0)$

Radial 3d Laplace's equation

If we assume the radial symmetry ansatz, the temperature $u(r, z)$ of the cylinder satisfies:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

We need to specify the boundary data:

bottom $u(r, 0)$

top $u(r, Z_0)$

side $u(R_0, z)$ or $\frac{\partial u}{\partial r}(R_0, z)$

Finding separable solutions to the radial 3d Laplace's equation

1. Separation gives $Z'' - \lambda Z = 0$, $rR'' + R' + r\lambda R = 0$
2. Dirichlet: $\lambda = \left(\frac{j_{0,i}}{R_0}\right)^2$; Neumann: $\lambda = \left(\frac{j_{1,i}}{R_0}\right)^2$
3. Recognise that

$$\sum_{i=1}^{\infty} \left[A_i \cosh\left(\alpha_i z\right) + B_i \sinh\left(\alpha_i z\right)\right] J_0\left(\alpha_i r\right)$$

   formally solves Laplace's equation.
4. Use the $n = 0$ Bessel series for the boundary data to find A_i and B_i.

Radial 2d heat equation

The 2-dimensional heat equation is:

$$\frac{\partial u}{\partial t} = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Homework:

1. Write the heat equation in polar coordinates (r, θ, t)
2. When is the radial symmetry ansatz reasonable for this equation?
3. What should the radial separable solutions look like?