Transform Methods I
textbook sections 15.2, 15.3

MATH 241

February 14, 2012

Definition

Given a function $f(x)$ defined for $x \geq 0$, the Laplace transform of f is

$$\mathscr{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt$$

Operational properties of the Laplace transform

linearity $\mathscr{L}\{af + bg\}(s) = a\mathscr{L}\{f\}(s) + b\mathscr{L}\{g\}(s)$

differentiation $\mathscr{L}\{f'\}(s) = -f(0) + s\mathscr{L}\{f\}(s)$

The Point

The Laplace transform turns statements about derivatives into statements about multiplication!

solving PDE with the Laplace transform

1. If the unknown function is $u(x, y)$, apply the Laplace transform to get $U(x, s)$.
2. Get an ODE for $U(x, s)$ in x with parameter s.
3. Solve for $U(x, s)$.
4. Try to think of some $u(x, y)$ with $\mathscr{L}\{u(x, y)\} = U(x, s)$.

Ay, there's the rub!

Fourier series (again)

If f is defined on $(-\infty, \infty)$, then for each $(-p, p)$ we have the Fourier series:

$$\frac{1}{2p}\int_{-p}^{p} f(t)dt + \frac{1}{p}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos\left(\frac{n\pi}{p}t\right)dt\right)\cos\left(\frac{n\pi}{p}x\right) + \left(\int_{-p}^{p} f(t)\sin\left(\frac{n\pi}{p}t\right)dt\right)\sin\left(\frac{n\pi}{p}x\right)\right]$$

$$\frac{1}{2\pi}\int_{-p}^{p} f(t)dt\frac{\pi}{p}$$
$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[A\left(\frac{n\pi}{p}\right)\cos\left(\frac{n\pi}{p}x\right)+B\left(\frac{n\pi}{p}\right)\sin\left(\frac{n\pi}{p}x\right)\right]\frac{\pi}{p}$$

$$\frac{1}{2\pi}\int_{-p}^{p} f(t)dt\frac{\pi}{p}$$
$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[A\left(\alpha_n\right)\cos\left(\alpha_n x\right)+B\left(\alpha_n\right)\sin\left(\alpha_n x\right)\right]\left(\alpha_{n+1}-\alpha_n\right)$$

The second term is a left-hand sum! As $p \to \infty$. . .

Definition

The Fourier integral of a function $f(x)$ defined on $(-\infty, \infty)$ is:

$$\frac{1}{\pi}\int_0^\infty (A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x))\, d\alpha$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(t)\cos(\alpha t) dt$$
$$B(\alpha) = \int_{-\infty}^{\infty} f(t)\sin(\alpha t) dt$$

$A(\alpha)$ is even.

$B(\alpha)$ is odd.

Theorem

If f and f' are piecewise continuous on every interval $[-p, p]$ and $\int_{-\infty}^{\infty} |f(t)| dt$ converges, then the Fourier integral converges to

- $f(x)$ *if f is continuous at x.*
- $\frac{1}{2}\left(\lim_{t \to x^+} f(t) + \lim_{t \to x^-} f(t)\right)$ *if f is not continuous at x.*

Theorem

If f is odd then its Fourier integral is

$$\frac{2}{\pi}\int_0^\infty (B(\alpha)\sin(\alpha x))\,d\alpha$$

where

$$B(\alpha) = \int_0^\infty f(t)\sin(\alpha t)dt$$

If f is even then its Fourier integral is

$$\frac{2}{\pi}\int_0^\infty (A\alpha)\cos(\alpha x))\,d\alpha$$

where

$$A(\alpha) = \int_0^\infty f(t)\cos(\alpha t)dt$$

Definition

The complex Fourier integral for f is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha)e^{-i\alpha x}\,d\alpha$$

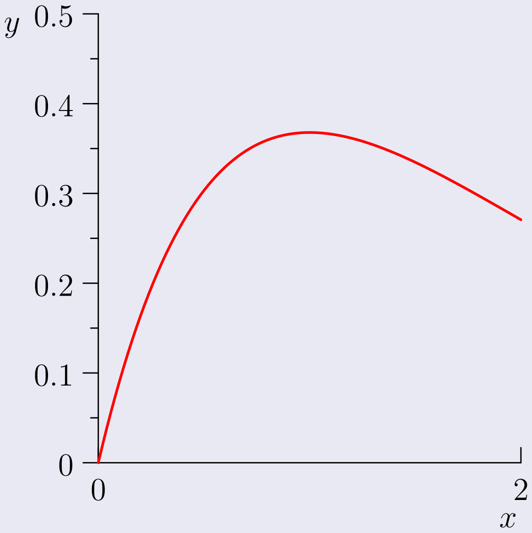
where

$$C(\alpha) = \int_{-\infty}^{\infty} f(t)e^{i\alpha t}\,dt$$

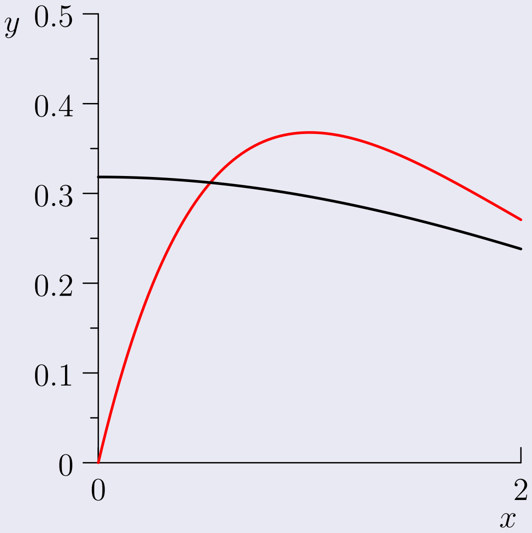
The Fourier integral coefficients are related by:

$$C(\alpha) = A(\alpha) + iB(\alpha)$$

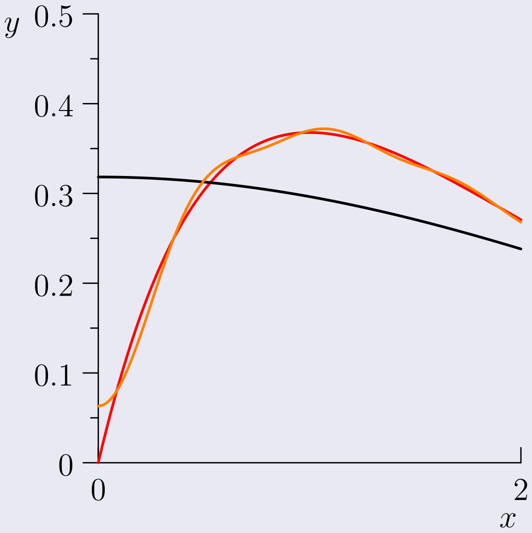
partial cosine integrals for xe^{-x}



partial cosine integrals for xe^{-x}



partial cosine integrals for xe^{-x}



partial cosine integrals for xe^{-x}