Transform Methods II
textbook sections 15.4

MATH 241

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Definition

Given a function $f(x)$ defined for all x, the **Fourier transform** of f is

$$\mathscr{F}\{f\}(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha t} f(t)dt = \hat{f}(\alpha)$$

This is just the complex Fourier integral!

Operational properties of the Fourier transform

linearity $\mathscr{F}\{af + bg\}(\alpha) = a\mathscr{F}\{f\}(\alpha) + b\mathscr{F}\{g\}(\alpha)$

differentiation $\mathscr{F}\{f'\}(\alpha) = -i\alpha\mathscr{F}\{f\}(\alpha)$

twice differentiation $\mathscr{F}\{f''\}(\alpha) = -\alpha^2\mathscr{F}\{f\}(\alpha)$

solving PDE with the Fourier transform

1. If the unknown function is $u(x, y)$, apply the Fourier transform to get $U(x, \alpha)$.
2. Get an ODE for $U(x, \alpha)$ in x with parameter α.
3. Solve for $U(x, \alpha)$.
4. Try to think of some $u(x, y)$ with $\mathscr{F}\{u(x, y)\} = U(x, \alpha)$.

Recall

If f is continuous at x, and its complex Fourier series is given by $C(\alpha)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\alpha) e^{-i\alpha x} d\alpha$$

so if we find $C(\alpha)$ we can find f!

Definition

The inverse Fourier transform of $F(\alpha)$ is

$$\mathscr{F}^{-1}\{F\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-ixs}ds = \check{F}(x)$$

Theorem

The inverse Fourier transform is the inverse of the Fourier transform, i.e.

$$\mathscr{F}^{-1}\{\mathscr{F}\{f\}\}(x) = f(x)$$
$$\check{\hat{f}}(x) = f(x)$$

Definition

A transform T is called integral if it is given by integrating against a function:

$$T\{f\}(\alpha) = \int f(t)K(\alpha, t)dt$$

The function $K(\alpha, t)$ is called the kernel of the transform.

Definition

The Fourier sine and Fourier cosine transforms are given by:

$$\mathscr{F}_s\{f\}(\alpha) = \int_0^\infty f(t)\sin(\alpha t)dt$$

$$\mathscr{F}_c\{f\}(\alpha) = \int_0^\infty f(t)\cos(\alpha t)dt$$

Theorem

The sine and cosine transforms have integral inverses:

$$\mathscr{F}_s^{-1}\{F\}(x) = \frac{2}{\pi}\int_0^\infty F(s)\sin(sx)ds$$

$$\mathscr{F}_c^{-1}\{F\}(x) = \frac{2}{\pi}\int_0^\infty F(s)\cos(sx)ds$$

The Fourier transforms are known as unitary.

Sine and cosine transforms and derivatives

$$\mathscr{F}_s\{f'\}(\alpha) = -\alpha\mathscr{F}_c\{f\}(\alpha)$$
$$\mathscr{F}_c\{f'\}(\alpha) = \alpha\mathscr{F}_s\{f\}(\alpha) - f(0)$$
$$\mathscr{F}_s\{f''\}(\alpha) = -\alpha^2\mathscr{F}_s\{f\}(\alpha) + \alpha f(0)$$
$$\mathscr{F}_c\{f''\}(\alpha) = -\alpha^2\mathscr{F}_c\{f\}(\alpha) - f'(0)$$