# Complex Numbers & Arithmetic
# textbook sections 17.1-17.2

MATH 241

February 21, 2012

The shortest path between two truths in the real domain passes through the complex domain.

*Jacques Hadamard*

**Theorem**

*Every quadratic equation* $ax^2 + bx + c = 0$*, where* $a, b, c$ *are real numbers and* $a \neq 0$*, has two solutions:*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## Definition

A complex number is $z = a + ib$ where $a, b$ are real and $i^2 = -1$.

$a$ is the real part and $b$ is the imaginary part:

$$a = \mathfrak{Re}(z) \qquad\qquad b = \mathfrak{Im}(z)$$

## Definition

Complex arithmetic is obtained by treating complex numbers as formal sums and remembering $i^2 = -1$.

## Complex arithmetic

If $z_1 = a + ib$ and $z_2 = x + iy$, then

$$z_1 + z_2 = (a + x) + i(b + y)$$

$$z_1 z_2 = ax - by + i(ay + bx)$$

$$\frac{z_1}{z_2} = \frac{ax + by}{x^2 + y^2} + i\frac{bx - ay}{x^2 + y^2}$$

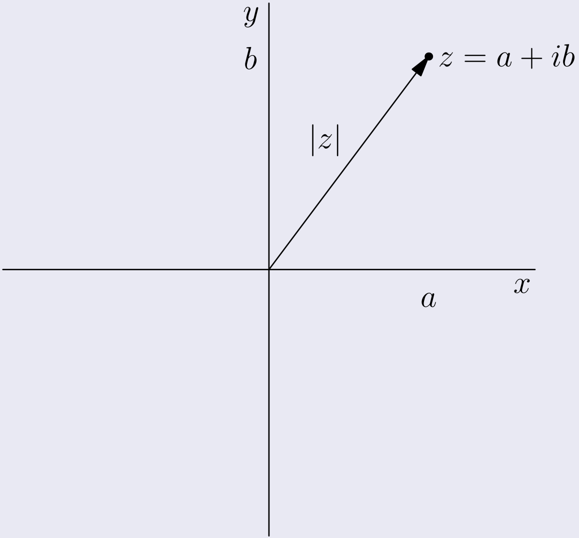
Don't memorise these formulas!

## Definition

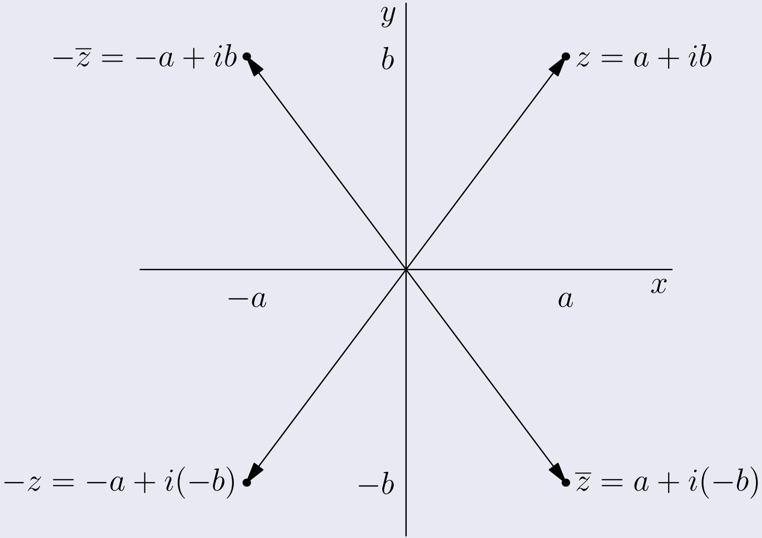
The modulus (or absolute value or norm) of a complex number $z = a + ib$ is

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$$

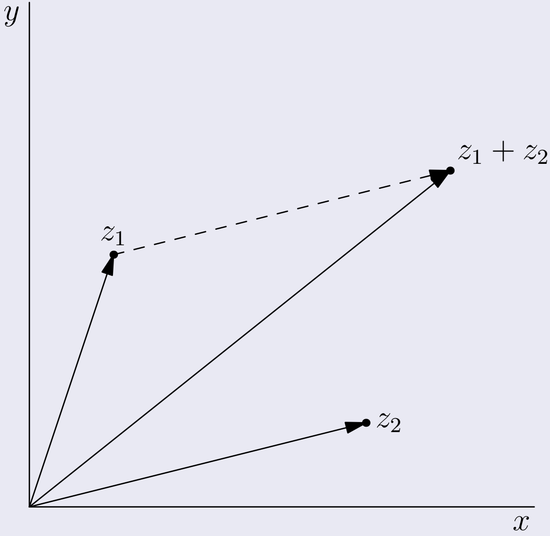
## Geometric interpretation



## Geometric interpretation



# Addition

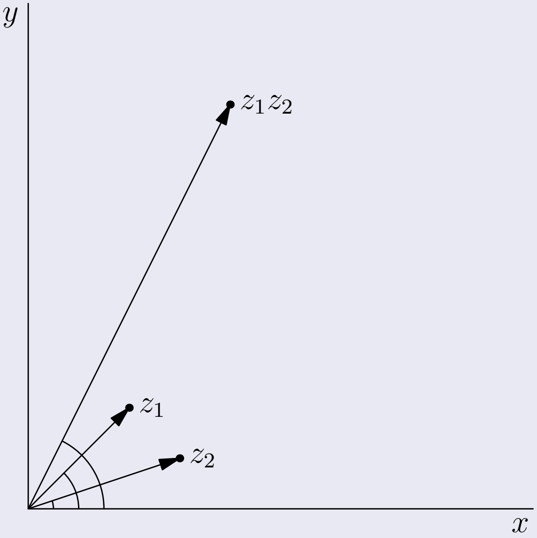


## Triangle Inequality

Given any complex numbers $z_1, z_2$, we have the inequalities:

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

## Multiplication



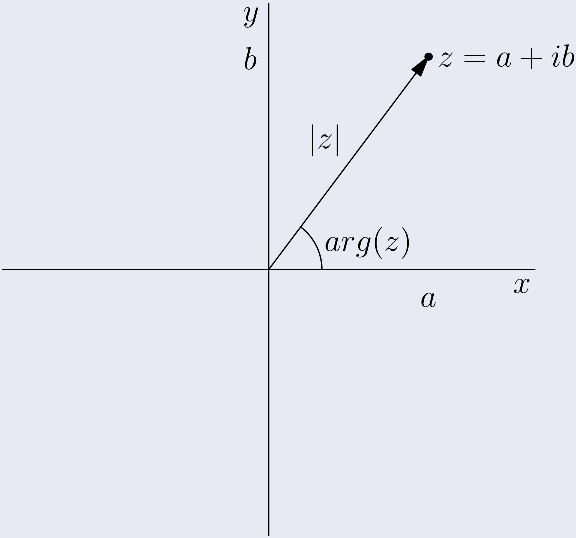
## Multiplication

Multiplication mutliplies moduli and adds angles.

## De Moivre's formula

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

## Geometric interpretation



## Arguments

For any two complex numbers $z_1, z_2$, we have:

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$
$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

but not necessarily

$$\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$$

## Roots

The complex number $z$ has exactly $n$ $n^{th}$ roots, given by:

$$|z|^{\frac{1}{n}}\left[\cos\left(\frac{\text{Arg}(z)}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\text{Arg}(z)}{n}+\frac{2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$.

The $n^{th}$ root given by $k = 0$ is called the principal $n^{th}$ root.