Complex Functions II
textbook section 17.4

MATH 241

February 28, 2012

Recall

An assignment to each point in the plane of a 2-vector is a vector field.

That's exactly what a complex function does!

The vector fields we might be interested in are velocity fields of incompressible fluids.

If we drop a particle into an incompressible fluid flowing by the vector field $\langle u, v \rangle$, its motion is given by:

$$\frac{dx}{dt} = u(x, y)$$
$$\frac{dy}{dt} = v(x, y)$$

The analytic properties of the function turn out to say a lot about the fluid.

Definition

The complex function f has the limit L at z_0, or

$$\lim_{z \longrightarrow z_0} f(z) = L$$

if for every $\epsilon > 0$ there is a $\delta > 0$ for which

$$0 < |z - z_0| < \delta$$

guarantees

$$|f(z) - L| < \epsilon$$

Note

Formally this is identical to the definition of limits for real functions! But every symbol means something different now.

Theorem

Let $f = u + iv$ *be a complex function,* $z_0 = x_0 + iy_0$, $L = A + iB$. *Then*

$$\lim_{z \to z_0} f(z) = L$$

if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = A$$

and

$$\lim_{(x,y) \to (x_0,y_0)} v(x,y) = B$$

Recall

The Two-Path Test says that if u (or v) has two different limits along two different paths into (x_0, y_0), then the limit of u (or v) at (x_0, y_0) does not exist!

Theorem

Limits respect arithmetic, i.e.

$$\lim_{z \longrightarrow z_0} (f(z) \pm g(z)) = \left(\lim_{z \longrightarrow z_0} f(z) \right) \pm \left(\lim_{z \longrightarrow z_0} g(z) \right)$$

$$\lim_{z \longrightarrow z_0} (f(z) g(z)) = \left(\lim_{z \longrightarrow z_0} f(z) \right) \left(\lim_{z \longrightarrow z_0} g(z) \right)$$

$$\lim_{z \longrightarrow z_0} |f(z)| = \left| \lim_{z \longrightarrow z_0} f(z) \right|$$

Definition

f is continuous at z_0 if $f(z_0) = \lim_{z \to z_0} f(z)$.

f is continuous on S if f is continuous at each point of S.

Corollary

Sums, products, and quotients of continuous functions are continuous.

Definition

The complex derivative of f at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided the limit exists.

If $f'(z_0)$ exists, we say f is differentiable at z_0.

Note

Again, this is *formally* the same as for real differentiation. But notice that

- The limit is a complex limit.
- The differential Δz is a complex number: $\Delta z = \Delta x + i\Delta y$

Theorem

All formal properties of the real derivative carry over.

Caution

That's just about all that carries over. Many functions which "look" complex-differentiable aren't.