Analyticity and the Cauchy-Riemann Equations textbook section 17.5

MATH 241

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Recall

$f(z)$ is differentiable at $z = z_0$ if the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists.

Definition

f is analytic at z_0 if f is differentiable in a neighbourhood of z_0.

f is analytic on the set S if f is analytic at every point of S.

f is entire if f is analytic on the entire plane.

Theorem

If $f(x+iy) = u(x,y) + iv(x,y)$ *is differentiable at* $z_0 = x_0 + iy_0$, *then* u *and* v *satisfy the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x}\Big|_{(x_0,y_0)} = \frac{\partial v}{\partial y}\Big|_{(x_0,y_0)}$$
$$\frac{\partial v}{\partial x}\Big|_{(x_0,y_0)} = -\frac{\partial u}{\partial y}\Big|_{(x_0,y_0)}$$

formulas for $f'(z)$

The Cauchy-Riemann equations give two useful formulas for $f'(z)$:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

a mnemonic

$$\frac{d}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

Theorem

Suppose u and v are continuous and have continuous first partials in a domain D. If u and v satisfy the Cauchy-Riemann equations in D, then $f = u + iv$ is analytic in D.

Theorem

If $f(x+iy) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v are harmonic, i.e. satisfy Laplace's equation:

$$\Delta u = 0$$
$$\Delta v = 0$$

Definition

If $u(x,y)$ is a function which is harmonic in a domain D, and $v(x,y)$ is another function on D so that

$$f(x+iy) = u(x,y) + iv(x,y)$$

is analytic in D, we call v a harmonic conjugate of u.