Series Representations I
textbook section 12.1

MATH 241

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Definition

Given $f(t)$ and $g(t)$ two functions defined on an interval $[a, b]$, the inner product of f and g on $[a, b]$ is

$$(f, g) = \int_a^b f(t)g(t)dt$$

f and g are orthogonal on the interval $[a, b]$ if $(f, g) = 0$.

Theorem

If f, g, h are functions on the interval $[a, b]$ and c is a real number, then

$$\begin{aligned} (f+g, h) &= (f, h) + (g, h) \\ (f, g+h) &= (f, g) + (f, h) \\ (cf, g) &= c(f, g) = (f, cg) \\ (f, f) &\geq 0 \end{aligned}$$

In the last inequality, $(f, f) = 0$ exactly when f is the constant function 0.

Definition

The norm of f on $[a, b]$ is

$$\|f\| = \sqrt{(f, f)}$$

Definition

An orthogonal set on $[a, b]$ is a set of functions $\{\phi_m\}$ defined on $[a, b]$ so that $(\phi_m, \phi_n) = 0$ whenever $m \neq n$.
An orthonormal set on $[a, b]$ is an orthogonal set with $\|\phi_m\| = 1$.

important e.g.g.

The following are orthogonal sets:

- $\{1, \cos(mx)\}, m = 1, 2, 3, \ldots$ *on* $[-\pi, \pi]$
- $\{1, \sin(mx)\}, m = 1, 2, 3, \ldots$ *on* $[-\pi, \pi]$
- $\{1, \cos(mx), \sin(nx)\}, m, n = 1, 2, 3, \ldots$ *on* $[-\pi, \pi]$

important e.g.g.

The following are orthonormal sets:

- $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(mx)}{\sqrt{\pi}} \right\}, m = 1, 2, 3, \ldots$ *on* $[-\pi, \pi]$
- $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin(mx)}{\sqrt{\pi}} \right\}, m = 1, 2, 3, \ldots$ *on* $[-\pi, \pi]$
- $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(mx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}} \right\}, m, n = 1, 2, 3, \ldots$ *on* $[-\pi, \pi]$

If $\{\phi_m\}$ is an orthogonal set on $[a, b]$, and $f(x)$ can be written as $f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots$, then

$$c_m = \frac{(f, \phi_m)}{\|\phi_m\|^2}$$

That is,

$$f(x) = \sum_{m=0}^{\infty} \frac{(f, \phi_m)}{\|\phi_m\|^2} \phi_m(x)$$

Definition

The inner product on $[a, b]$ with weight function w is

$$(f, g)_w = \int_a^b f(t)g(t)w(t)dt$$

Two functions f, g are orthogonal on $[a, b]$ with respect to the weight function w if $(f, g)_w = 0$.

If $\{\phi_m\}$ are orthogonal with respect to w and $f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots$, then

$$c_m = \frac{(f, \phi_m)_w}{(\phi_m, \phi_m)_w}$$

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Definition

The orthogonal set $\{\phi_m\}$ is called complete if the only continuous function on $[a, b]$ which is orthogonal to every $\{\phi_m\}$ is the constant function 0.

Telling whether an orthogonal set is complete

. . . is beyond the scope of this course.