Elementary Complex Functions I
textbook sections 17.6-17.7

MATH 241

March 15, 2012

Elementary Real Functions

- polynomials $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
- exponential functions e^x, b^x
- trigonometric functions $\sin x$, $\cos x$
- hyperbolic functions $\sinh x = \frac{1}{2}(e^x - e^{-x})$, $\cosh x = \frac{1}{2}(e^x + e^{-x})$
- logarithmic functions $\ln x$
- inverse trigonometric functions $\arcsin x$, $\arccos x$

These are all continuous and differentiable to all orders.

Complex polynomials

Complex polynomials:

$$\alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0$$

where α_i are complex numbers and z is a complex variable

Complex exponential

Definition

The real exponential function $g(x) = e^x$ is the unique solution of the IVP $g' = g, g(0) = 1$.

The function $f(x + iy) = e^x \cos y + ie^x \sin y$ has $\frac{d}{dz}f = f$ and $f(0) = 1$.

Definition

The complex exponential function is given by:

$$\exp(x + iy) = e^x \cos y + ie^x \sin y$$

Complex exponential

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- $\exp(z + w) = \exp(z)\exp(w)$

Complex exponential

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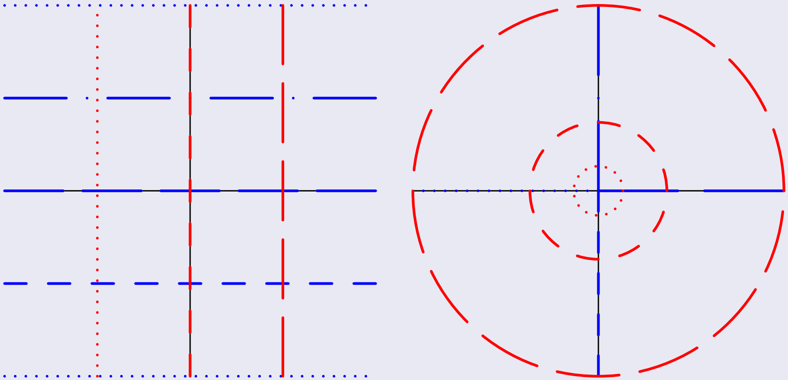
- If $z = 0 + iy$ is purely imaginary, then $\exp(z) = e^{iy} = \cos(y) + i\sin(y)$
- $\exp(z + w) = \exp(z)\exp(w)$
- $\exp(z + 2\pi i) = \exp(z)$

Complex exponential function



Complex logarithm

Recall

If x and y are real numbers,

$$e^y = x$$
$$y = \ln x$$

Since e^y is invertible, $\ln x$ is a function.

If we set $\exp(w) = z$ and solve for w, we get

$$w = \ln|z| + i\arg(z)$$

Note

$\ln z$ is not a function!

Complex logarithm

Definition

The principal logarithm of z is

$$\mathrm{Ln}(z) = \ln|z| + i\mathrm{Arg}(z)$$

$\mathrm{Ln} z$ is analytic except where $\arg(z) = \pi$.

Other exponentials

Definition

If b is a complex number, we define

$$b^z = \exp(z \ln b)$$

Note

b^z may not be a function.

Definition

The principal value of b^z is $\exp(z\mathrm{Ln}b)$.

Trigonometric functions

Definition

The real trigonometric functions $c(x) = \cos(x)$ and $s(x) = \sin(x)$ are the unique solution of the IVP

$$\begin{pmatrix} s' \\ c' \end{pmatrix} = \begin{pmatrix} c \\ -s \end{pmatrix}, \begin{pmatrix} s(0) \\ c(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Definition

The complex trigonometric functions are:

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$$
$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$$

Trigonometric functions

Euler's formulae

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$
$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$
$$e^{iz} = \cos(z) + i\sin(z)$$

Trigonometric functions

Zeros

$\sin z = 0$ when $z = n\pi$

$\cos z = 0$ when $z = \frac{\pi}{2} + n\pi$

The trigonmetric functions are 2π-periodic.

$\sin(z)$

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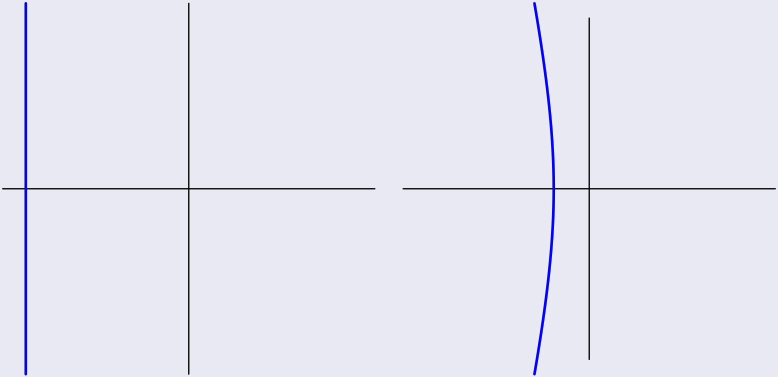

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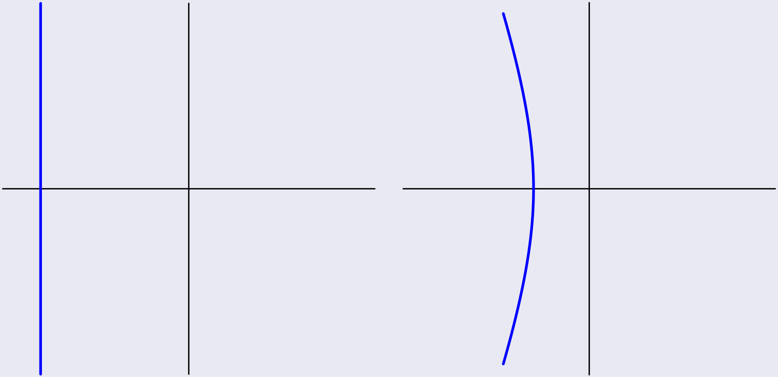

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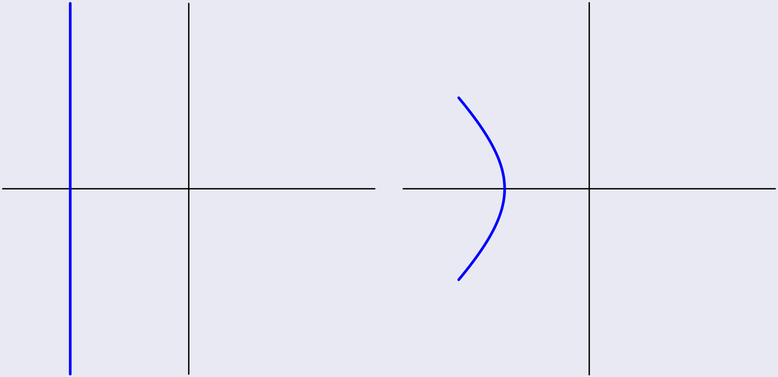

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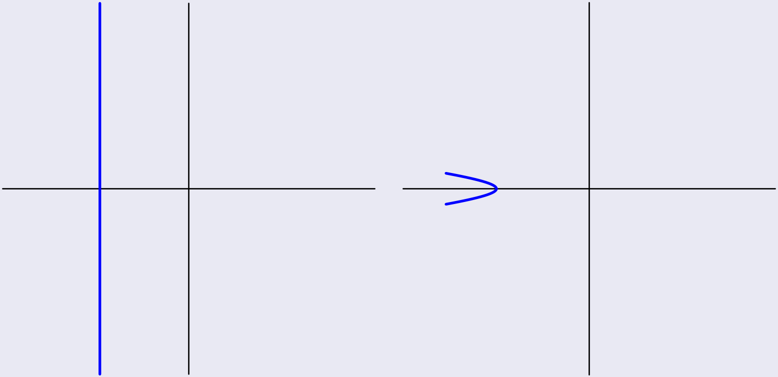

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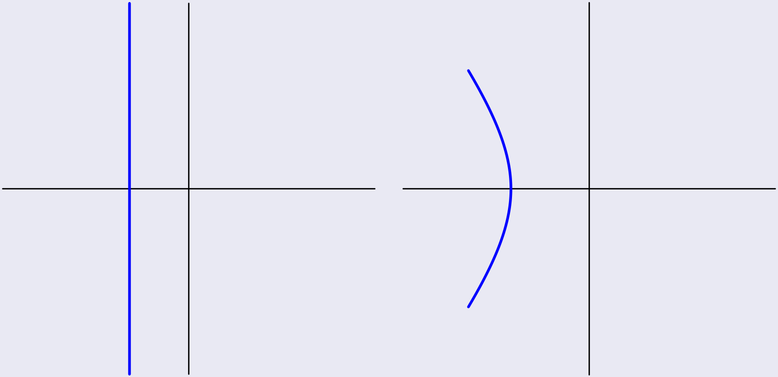

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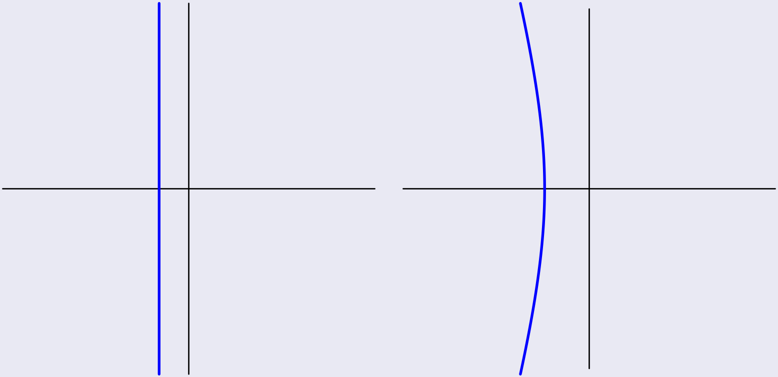

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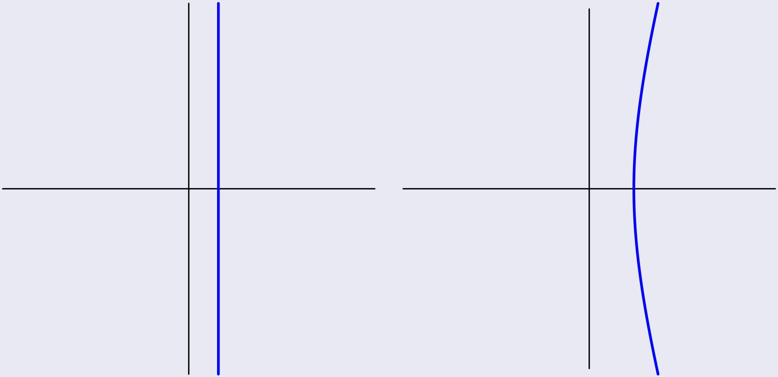

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