Contour Integrals I
textbook sections 18.1-18.2

MATH 241

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Recall

Given a real function of two real variables $g(x, y)$ and a smooth path in the plane C, with paramerisation $(x(t), y(t)), a \leq t \leq b$, we can define the line integral:

$$\int_C g ds = \int_a^b g(x(t), y(t)) \frac{ds}{dt} dt$$

where s is the arclength parameter.

$\frac{ds}{dt} = |v| = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$, so this is just

$$\int_a^b g(x(t), y(t)) \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} dt$$

Definition

A smooth contour is a $z(t) = x(t) + iy(t)$, where $(x(t), y(t))$ parametrises a smooth curve, i.e.

1. $x(t)$ and $y(t)$ are differentiable, and their derivatives are continuous.
2. $\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$ is never 0.

Given a smooth contour parametrised by $z(t)$, $a \leq t \leq b$, we write $\frac{dz}{dt} = z'(t) = \frac{dx}{dt} + i\frac{dy}{dt}$ and require $z' \neq 0$.

Definition

A piecewise-smooth contour is a finite collection of smooth contours joined end to end.

Definition

Given a complex function $f(z)$ continuous around* a piecewise-smooth contour C, we define the contour integral

$$\int_C f(z)dz = \lim \sum_{k=1}^{n} f(w_k)(z_k - z_{k-1})$$

where $z(a) = z_0, z(b) = z_n$, and $z_1, \ldots, z_{n-1}$ are points along C, with w_k a point on C between z_{k-1} and z_k, and the limit is taken as the distances between any two z_{k-1}, z_k go to zero.

Note

Each of these Riemann sums is a complex number, so $\int_c f(z)dz$ is a complex number.

Formulae for $\int_C f(z)dz$

If $f = u + iv$ and C is parametrised by $z(t)$, $a \leq t \leq b$, then

$$\int_C f(z)dz = \int_C (udx - vdy) + i \int_C (vdx + udy)$$

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Formal properties of the contour integral

Provided all integrals in sight make sense,

$$\int_C (f(z) + g(z))dz = \int_C f(z)dz + \int_C g(z)dz$$

$$\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

where $+$ means concatenation and $-$ means reversal.

Theorem

If C has length L and $|f(z)| \leq M$ along C, then

$$|\int_C f(z)dz| \leq ML$$

WARNING: SIMPLE CLOSED CURVES AHEAD

Recall

The circulation of the field $\mathbf{F} = \langle M, N \rangle$ around the simple closed contour C is:

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C M dx + N dy$$

The flux of F across C is:

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_C M dy - N dx$$

Reason to Use Contour Integrals

If C is a simple closed countour, then

$$\int_C \overline{f(z)} dz = (\text{circulation of } f) + i\,(\text{flux of } f)$$

Theorem (Cauchy-Goursat version 1)

If f is analytic along a simple closed contour C, and also analytic in the interior of C, then

$$\int_C f(z)dz = 0$$

Theorem (Cauchy-Goursat version 2)

If f is analytic in the simply-connected domain D, and C is a simple closed contour in D, then

$$\int_C f(z)dz = 0$$