Contour Integrals II
textbook sections 18.2-18.3

MATH 241

March 27, 2012

Recall

A domain D is simply connected if any loop in D can be continuously deformed to a point.

Definition

A domain which is not simply-connected is multiply-connected.

A domain with 1 hole is doubly-connected, a domain with two holes is triply-connected, etc.

Theorem (Cauchy-Goursat for simple closed curves)

If f is analytic along a simple closed contour C, and also analytic in the interior of C, then

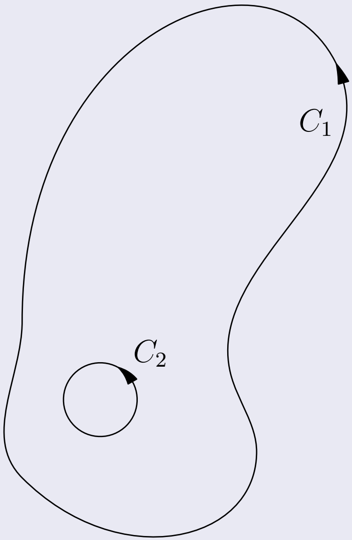
$$\int_C f(z)dz = 0$$

Theorem (Cauchy-Goursat for simply-connected domains)

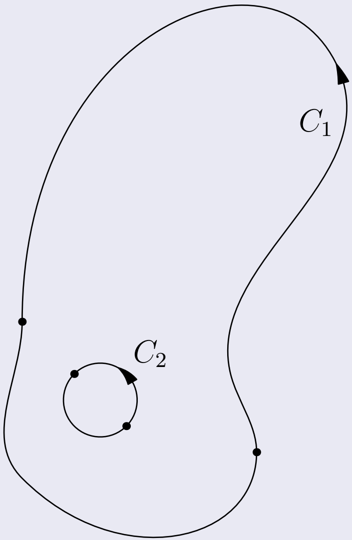
If f is analytic in the simply-connected domain D, and C is a simple closed contour in D, then

$$\int_C f(z)dz = 0$$

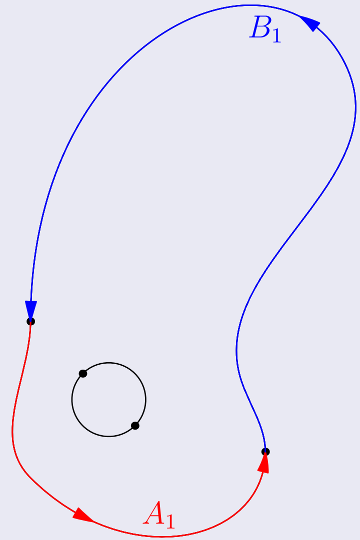
Deformation of Contours



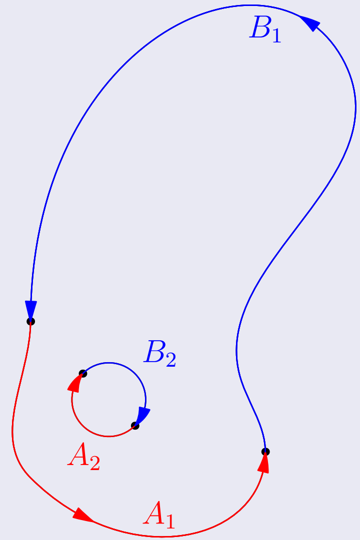
Deformation of Contours



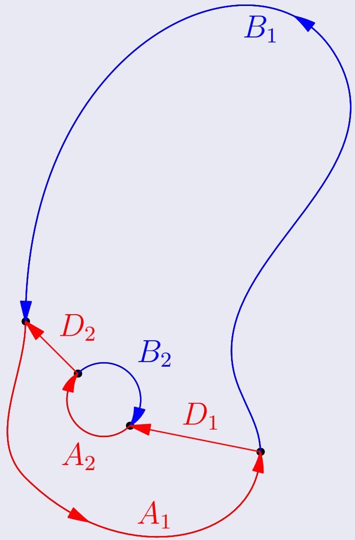
Deformation of Contours



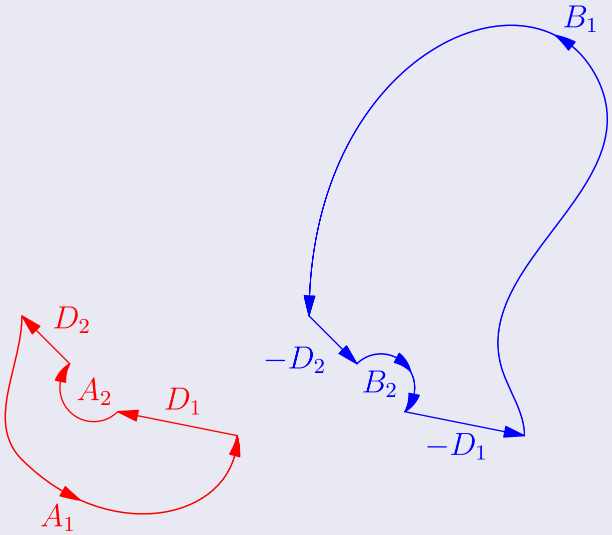
Deformation of Contours



Deformation of Contours



Deformation of Contours



Deformation of Contours

If f is analytic in the region bounded by C_1 and C_2, then

$$\int_{A_1+D_1+A_2+D_2} f(z)dz = 0$$

$$\int_{B_1-D_2+B_2-D_2} f(z)dz = 0$$

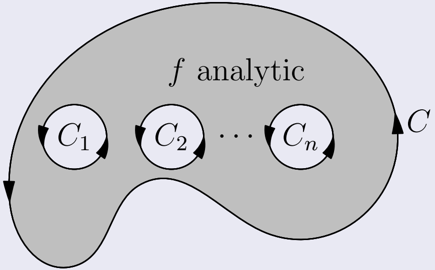
So

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

Theorem (Cauchy-Goursat for multiply-connected domains)

Suppose C is a simple closed contour, and $C_1, \ldots, C_n$ are nonintersecting simple closed contours which lie in the interior of C. If $f(z)$ is analytic on the region bounded by $C, C_1, \ldots, C_n$, then

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \cdots + \oint_{C_n} f(z)dz$$



Useful computation

If C_0 is a simple closed contour with z_0 inside C_0, then

$$\oint_{C_0} \frac{dz}{z - z_0} = 2\pi i$$

If $n \neq 1$, then

$$\oint_{C_0} \frac{dz}{(z - z_0)^n} = 0$$

Theorem (Cauchy-Goursat for nonsimple contours)

Suppose D is a simply-connected domain, C is a closed contour in D, and f is analytic in D. Then

$$\int_C f(z)dz = 0$$

Theorem (Fundamental Theorem of Contour Integration)

Suppose there is some $F(z)$ with $F'(z) = f(z)$ for all z in the domain D. Then if C is a contour in D with endpoints z_0, z_1,

$$\int_C f(z)dz = F(z_1) - F(z_0)$$

Theorem

If D is a simply-connected domain and f is analytic in D, then f has an antiderivative in D, i.e. there is $F(z)$ defined on D with $\frac{d}{dz}F = f$.

Note

The antiderivative F is itself analytic in D (since $F'(z)$ exists in a domain), so F has an antiderivative as well.