Sequences and Series
textbook section 19.1

MATH 241

April 5, 2012

Definition

A sequence of complex numbers $\{z_k\}$ is a choice of complex number for each natural number k.

Definition

The sequence $\{z_k\}$ converges to z if for every $\epsilon > 0$ there is N a natural number so that $k \geq N$ guarantees $|z - z_k| < \epsilon$.

Theorem

$z_k \to z$ *exactly if* $\mathfrak{Re}(z_k) \to \mathfrak{Re}(z)$ *AND* $\mathfrak{Im}(z_k) \to \mathfrak{Im}(z)$

$z_k \to 0$ *exactly if* $|z_k| \to 0$

If $z \neq 0$, *then* $z_k \to z$ *exactly if* $|z_k| \to |z|$ *AND* $\arg(z_k) \to \arg(z)$.

In other words

$\{z_k\}$ is convergent exactly if the coordinates of z_k are convergent as real sequences.

Definition

If $\{z_k\}$ are a sequence, the partial sums of $\{z_k\}$ are $S_n = z_1 + \cdots + z_n$.

If the sequence of partial sums $\{S_n\}$ converges to S, we write

$$\sum_{k=1}^{\infty} z_k = S$$

and say the series $\sum_{k=1}^{\infty} z_k$ converges to S. If the $\{S_n\}$ diverge, we say that the series $\sum_{k=1}^{\infty} z_k$ diverges.

Definition

Given a sequence $\{z_k\}$, consider the real series

$$\sum_{k=1}^{\infty} |z_k|$$

If this series converges, we say that the complex series $\sum_{k=1}^{\infty} z_k$ converges absolutely.

Theorem

If $\sum_{k=1}^{\infty} z_k$ *converges absolutely, then* $\sum_{k=1}^{\infty} z_k$ *converges.*

Important e.g.

The geometric series

$$\sum_{k=0}^{\infty} z^k$$

converges to $\frac{1}{1-z}$ if $|z| < 1$ and diverges if $|z| \geq 1$.

Theorem (Ratio Test)

Given a series $\sum_{k=1}^{\infty} z_k$, *let* $L = \lim \left| \frac{z_{k+1}}{z_k} \right|$.

- *If* $L < 1$, *then the series converges absolutely.*
- *If* $L > 1$, *then the series diverges.*
- *If* $L = 1$, *we don't know.*

Theorem (Root Test)

Given a series $\sum_{k=1}^{\infty} z_k$, *let* $L = \lim \sqrt[k]{|z_k|}$.

- *If* $L < 1$, *then the series converges absolutely.*
- *If* $L > 1$, *then the series diverges.*
- *If* $L = 1$, *we don't know.*

Note that the Ratio Test and the Root Test are both about sequences of real numbers.

Definition

A series of the form

$$\sum_{k=0}^{\infty} a_k \left(z - z_0\right)^k$$

is called a power series in z with centre z_0 and coefficients $\{a_k\}$.

Theorem (radius of convergence via Ratio Test)

Given a power series

$$\sum_{k=0}^{\infty} a_k \left(z - z_0\right)^k$$

let $\alpha = \lim \left|\frac{a_{k+1}}{a_k}\right|$.
The power series converges absolutely for all z *with* $|z - z_0| < \frac{1}{\alpha}$ *and diverges for all* z *with* $|z - z_0| > \frac{1}{\alpha}$.

Theorem (radius of convergence via Root Test)

Given a power series

$$\sum_{k=0}^{\infty} a_k \left(z - z_0\right)^k$$

let $\alpha = \lim \sqrt[k]{|a_k|}$.
The power series converges absolutely for all z *with* $|z - z_0| < \frac{1}{\alpha}$ *and diverges for all* z *with* $|z - z_0| > \frac{1}{\alpha}$.