# Taylor and Laurent Series
# textbook sections 19.2-19.3

MATH 241

April 10, 2012

## Definition

If $f$ is analytic at $z_0$, its Taylor series at $z = z_0$ is

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(z_0)\,(z - z_0)^k$$

## Theorem

*Within its radius of convergence, a power series:*

1. *is continuous.*
2. *is differentiable.*
3. *has derivative*

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$$

## Corollary

*A power series is analytic within its radius of convergence.*

## Theorem

*If $C$ is a contour in the disc of convergence, then*

$$\int_C \sum_{k=0}^{\infty} a_k (z - z_0)^k \, dz = \sum_{k=0}^{\infty} a_k \int_C (z - z_0)^k \, dz$$

**Theorem (Taylor's Theorem)**

*If $f(z)$ is an analytic function in the domain $D$, and $z_0$ is a point of $D$, then the series representation*

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(z_0)(z - z_0)^k$$

*is valid on any disc centred at $z_0$ which lies inside $D$.*

To find the radius of convergence for the Taylor series of $f$, compute the distance to the nearest point where $f$ is not analytic.

## Important Taylor series

$$\exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} z^{2k+1}$$

$$\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} z^{2k}$$

These are valid for all $z$.

## Important Taylor series

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

This is valid for $|z| < 1$.

$$\mathrm{Ln}(z) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{k} (z-1)^k$$

This is valid for $|z-1| < 1$.

## Definition

A series of the form

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

is called the Laurent series with centre $z_0$ and coefficients $\{a_k\}$.

The analytic part of the Laurent series is

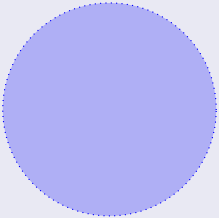
$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

The principal part of the Laurent series is

$$\sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k}$$

## Note

We expect a power series to converge on a disc: $|z - z_0| < R$.



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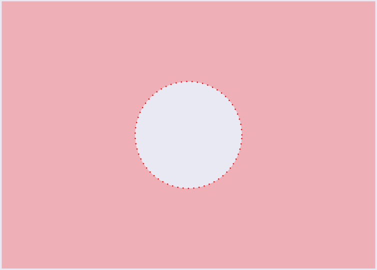

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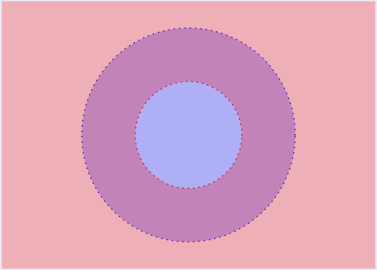

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## Theorem (Laurent's Theorem)

*Suppose $f(z)$ is analytic in the annulus $r < |z - z_0| < R$. Then for any $z$ in the annulus, the Laurent series representation is valid:*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

*where*

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{k+1}} dw$$

*for any contour $C$ which is equivalent to the boundary circles $|z - z_0| = R, |z - z_0| = r$.*