Laurent Series, Singularities, and Integration I textbook sections 19.3-19.5

MATH 241

April 12, 2012

Theorem (Laurent's Theorem)

Suppose $f(z)$ is analytic in the annulus $r < |z - z_0| < R$. Then for any z in the annulus, the Laurent series representation is valid:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{k+1}} dw$$

for any contour C which is equivalent to the boundary circles $|z - z_0| = R, |z - z_0| = r$.

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Theorem

$$\oint_C (z - z_0)^n \, dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Definition

z_0 is an isolated singularity of $f(z)$ if f fails to be analytic at z_0 but is analytic in some small punctured disc $0 < |z - z_0| < R$ around z_0.

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Computation

If C is a simple closed curve which contains exactly one isolated singularity in its interior, then

$$\oint_C f(z)dz = 2\pi i \operatorname{Res}(f; z_0)$$

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1. If $a_k = 0$ for all $k < 0$, then the singularity is removable.
2. If $a_{-1} \neq 0$ and $a_k = 0$ for all $k < -1$, then the singularity is a simple pole.
3. If $a_{-n} \neq 0$ and $a_k = 0$ for all $k < -n$, then the singularity is a pole of order n.
4. If infinitely many a_k with $k < 0$ are nonzero, then the singularity is essential.

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Definition

An analytic function f has a zero of order n at z_0 if $f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$.

Observation

If $f(z) = a_{-n}(z - z_0)^{-n} + \cdots + a_{-1}(z - z_0)^{-1} + \cdots$ has a pole of order n, then $(z - z_0)^n f(z)$ is analytic at z_0.

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Theorem

If f and g are analytic at z_0, $f(z_0) \neq 0$, and g has an isolated zero of order n at z_0, then $F(z) = \frac{f(z)}{g(z)}$ has a pole of order n at z_0.