Laurent Series, Singularities, and Integration II textbook sections 19.3-19.5

MATH 241

April 17, 2012

Definition

The residue of f at the isolated singularity z_0 is the coefficient a_{-1} in its Laurent expansion at z_0.

How to compute $\text{Res}(f; z_0)$:

- Compute a Laurent series centred at z_0.
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Theorem

If f has a simple pole at z_0, then

$$\text{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

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$$\text{Res}(f; z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right]$$

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Theorem

If $f(z)$ and $g(z)$ are analytic in a neighbourhood of z_0, $g(z)$ has a simple zero at z_0 and $f(z_0) \neq 0$, then

$$\text{Res}\left(\frac{f}{g}; z_0\right) = \frac{f(z_0)}{g'(z_0)}$$

Theorem (L'Hôpital's Rule)

If the limit

$$\lim_{z \longrightarrow z_0} \frac{f(z)}{g(z)}$$

has indeterminate form, then

$$\lim_{z \longrightarrow z_0} \frac{f(z)}{g(z)} = \lim_{z \longrightarrow z_0} \frac{f'(z)}{g'(z)}$$

How to compute $\text{Res}(f; z_0)$:

- Compute a Laurent series centred at z_0.
- Multiply by $(z - z_0)^n$ and take an appropriate limit.
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Theorem (Residue Theorem)

If f is a function which is analytic inside a simple closed curve C, except at finitely many isolated singular points $\{z_0, \ldots, z_n\}$, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=0}^{n} \text{Res}(f; z_k)$$

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Actually

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- Compute a Laurent series centred at z_0.
- Multiply by $(z - z_0)^n$ and take an appropriate limit.
- Use the Residue Theorem.

Actually

We'll primarily use the Residue Theorem in the following way:

1. Use the Residue Theorem to write an integral as a sum of residues.
2. Use the theorems given above to compute each residue.
3. Add the residues together to get the value of the integral.