Real Integrals
textbook section 19.6

MATH 241

April 19, 2012

Recall

If $h(x)$ is a real function defined for all real numbers x, then

$$\int_{-\infty}^{\infty} h(x)dx = \lim_{a \to -\infty} \int_{a}^{c} h(x)dx + \lim_{b \to \infty} \int_{c}^{b} h(x)dx$$

Definition

The Cauchy principal value of $\int_{-\infty}^{\infty} h(x)dx$ is

$$\text{P.V.} \int_{-\infty}^{\infty} h(x)dx = \lim_{R \to \infty} \int_{-R}^{R} h(x)dx$$

Fact

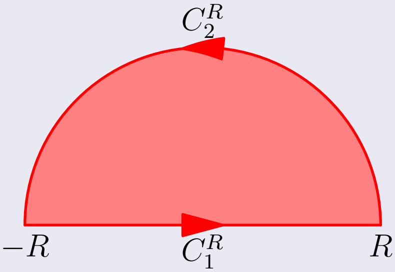
If $\int_{-\infty}^{\infty} h(x)dx$ exists, then $\int_{-\infty}^{\infty} h(x)dx = \text{P.V.} \int_{-\infty}^{\infty} h(x)dx$.

But $\text{P.V.} \int_{-\infty}^{\infty} h(x)dx$ can exist even if $\int_{-\infty}^{\infty} h(x)dx$ does not.

Use contour integrals to compute P.V.s

If the integrand is a rational function $h(x) = \frac{P(x)}{Q(x)}$, and $Q(x)$ is never zero, just extend to the corresponding complex rational function $f(z) = \frac{P(z)}{Q(z)}$.

Use the contour:



Computation

If P and Q are polynomials with $\deg(P) \leq \deg(Q) - 2$, then

$$\lim_{R\to\infty} \int_{C_2^R} \frac{P(z)}{Q(z)} dz = 0$$

Computation

If P and Q are polynomials with $\deg(P) \leq \deg(Q) - 2$, Q has no real zeros, and $\{z_1, \cdots, z_p\}$ are the zeros of Q which lie in the upper half-plane, then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{k=1}^{p} \text{Res}\left(\frac{P(z)}{Q(z)}; z_k\right)$$

Important note

The integral of a real-valued function must be real.

If we use residues to compute a real integral and get a complex number which is not real, *something is wrong*.

Fourier transform-type integrals

If we want to compute

$$\text{P.V.} \int_{-\infty}^{\infty} \sin(\alpha x) \frac{P(x)}{Q(x)} dx$$

or

$$\text{P.V.} \int_{-\infty}^{\infty} \cos(\alpha x) \frac{P(x)}{Q(x)} dx$$

We use the function

$$f(z) = \exp(i\alpha z) \frac{P(z)}{Q(z)}$$

and the same contour $C_1^R + C_2^R$ as before.

Computation

If P and Q are polynomials with $\deg(P) \leq \deg(Q) - 1$, and $\alpha > 0$, then

$$\lim_{R\to\infty} \int_{C_2^R} \exp(i\alpha z) \frac{P(z)}{Q(z)} dz = 0$$

Computation

If P and Q are polynomials with $\deg(P) \leq \deg(Q) - 1$, $\alpha > 0$, Q has no real zeros, and $\{z_1, \cdots, z_p\}$ are the zeros of Q which lie in the upper half-plane, then

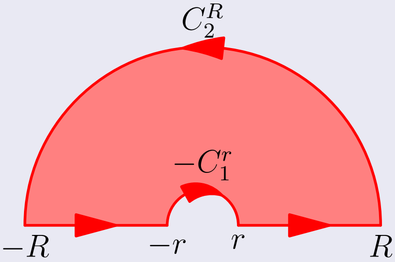
$$\text{P.V.} \int_{-\infty}^{\infty} \cos(\alpha x) \frac{P(x)}{Q(x)} dx = \mathfrak{Re}\left(2\pi i \sum_{k=1}^{p} \text{Res}\left(\exp(i\alpha z)\frac{P(z)}{Q(z)}; z_k\right)\right)$$

$$\text{P.V.} \int_{-\infty}^{\infty} \sin(\alpha x) \frac{P(x)}{Q(x)} dx = \mathfrak{Im}\left(2\pi i \sum_{k=1}^{p} \text{Res}\left(\exp(i\alpha z)\frac{P(z)}{Q(z)}; z_k\right)\right)$$

If $h(x)$ is defined for all real numbers except x_0, then

$$\text{P.V.} \int_{-\infty}^{\infty} h(x)dx = \lim_{R\to\infty} \lim_{r\to 0} \int_{-R}^{x_0-r} h(x)dx + \int_{x_0+r}^{R} h(x)dx$$

To find P. V. $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ if Q has real zeros, use the indented contour



Computation

If $f(z)$ has a simple pole at z_0 and C_1^r is the upper half of the circle of radius r centred at z_0, then

$$\lim_{r \to 0} \int_{C_1^r} f(z)dz = \pi i \operatorname{Res}(f; z_0)$$

Trigonometric integrals

We can also use contour integration to find integrals of the form

$$\int_0^{2\pi} F(\cos(\theta), \sin(\theta))d\theta$$

First make the substitution $z = e^{i\theta}$. Since $0 \leq \theta \leq 2\pi$, z traces out the unit circle. We have the following substitutions:

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$
$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$d\theta = -i\frac{dz}{z}$$

Trigonometric integrals

So

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta = -i \oint_{|z|=1} \frac{F(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z}))}{z} dz$$