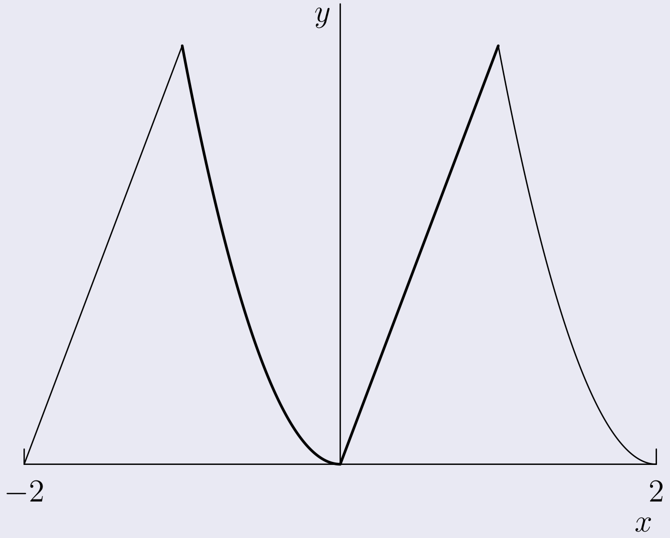
Complex Fourier Series
textbook section 12.4

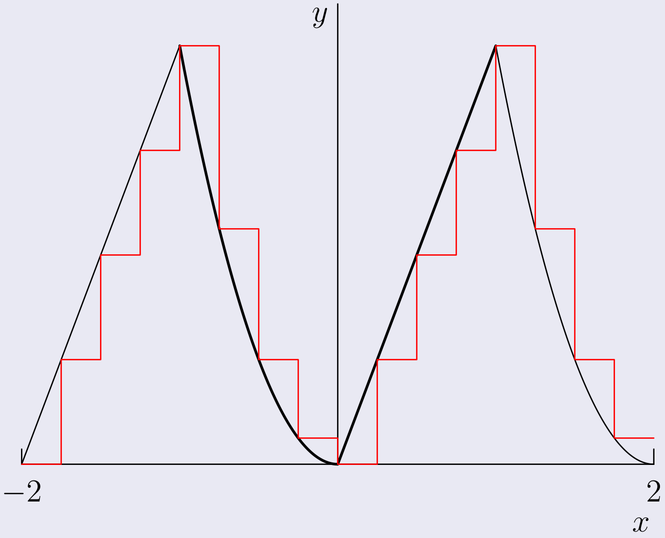
MATH 241

January 19, 2012

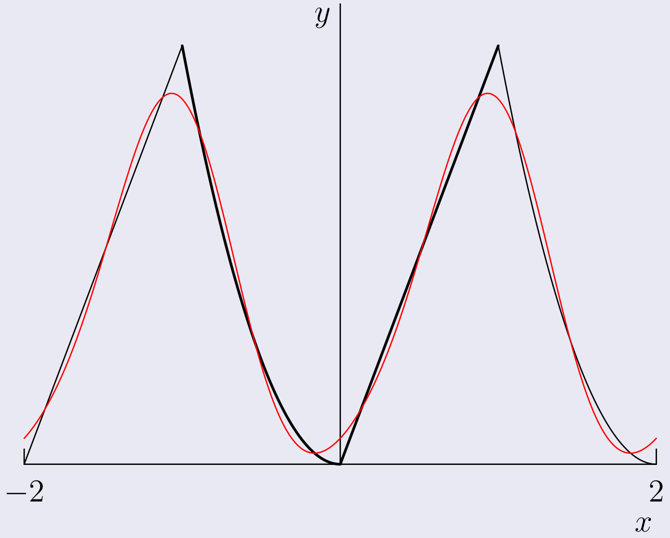
signal compression with bandwidth=8, base signal



signal compression with bandwidth=8, sampling



signal compression with bandwidth=8, Fourier



Note

The frequencies of $\cos\left(\frac{n\pi}{p}x\right)$ and $\sin\left(\frac{n\pi}{p}x\right)$ are the same! (So we can buy the same model oscillator to account for both of them.)

Question

How many of each frequency of oscillator do we need?

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- $z\overline{z} = |z|^2$

Euler's Formula

If x is a real number, then

$$e^{ix} = \cos(x) + i\sin(x)$$

Euler's Other Formulae

If x is a real number, then

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$
$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

Definition

The inner product of two complex-valued functions $F(x)$ and $G(x)$ on the interval $[a, b]$ is:

$$(F, G) = \int_a^b F(x)\overline{G(x)}dx$$

Complex Fourier Basis

The set

$$\{1, e^{\frac{i\pi}{p}x}, e^{\frac{2i\pi}{p}x}, \ldots, e^{-\frac{i\pi}{p}x}, e^{-\frac{2i\pi}{p}x}, \ldots\}$$

is orthogonal on $[-p, p]$.
In fact it's a complete orthogonal set for complex-valued functions on $[-p, p]$.

Complex Fourier Coefficients

The complex Fourier series for a function $f(x)$ defined on $[-p, p]$ is

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{p}x}$$

where

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-\frac{in\pi}{p}x}dx$$

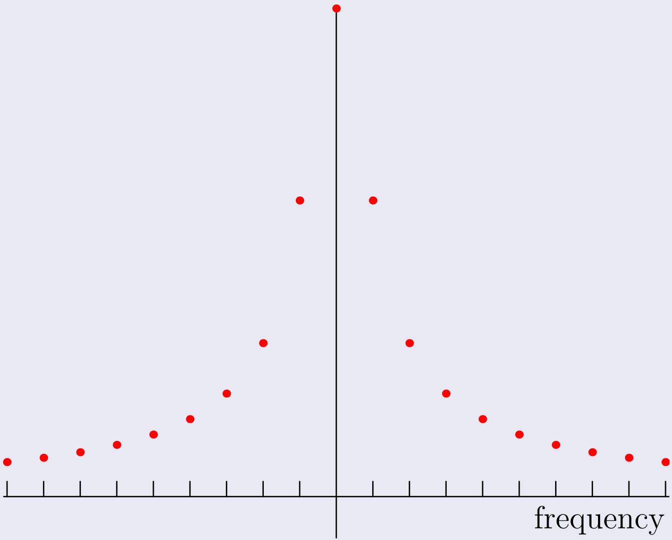
If $f(x)$ is real-valued, the complex Fourier coefficients are related to the real Fourier coefficients.

$$c_0 = \frac{1}{2}a_0$$
$$c_n = \frac{1}{2}(a_n - ib_n)$$
$$c_{-n} = \frac{1}{2}(a_n + ib_n)$$

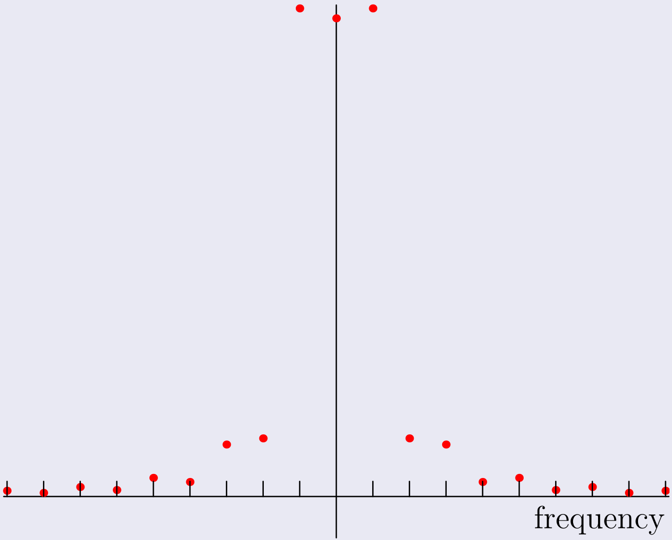
Definition

The frequency spectrum of a periodic function with period $T = 2p$ is the plot of $\left(\frac{n\pi}{p}, |c_n|\right)$.

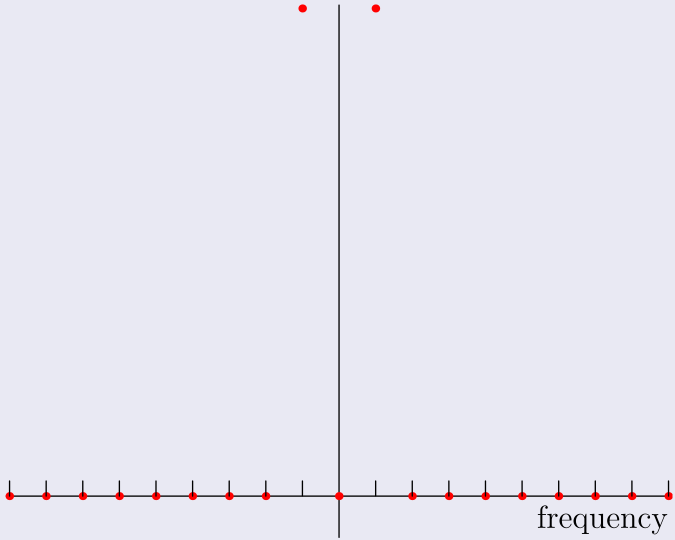
frequency spectrum for $f(x) = e^x, -1 \leq x \leq 1$



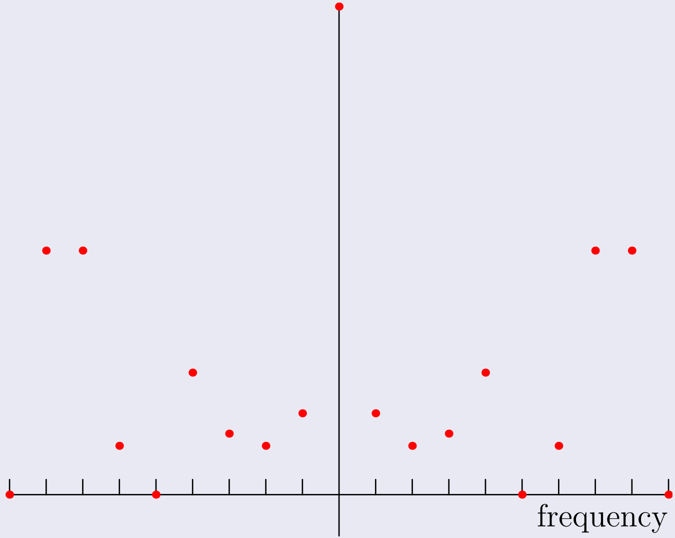
frequency spectrum for $f(x) = \begin{cases} x^2 & \\ x & \end{cases}, -1 \leq x \leq 1$



frequency spectrum for $f(x) = \sin(\pi x), -1 \le x \le 1$



frequency spectrum



frequency spectrum