Sturm-Liouville Problems
textbook section 12.5

MATH 241

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Liouville
Sluggers

Definition

The boundary-value problem

$$\begin{cases} y'' + \lambda y & = 0 \\ y(0) & = 0 \\ y(L) & = 0 \end{cases}$$

is called the Dirichlet eigenvalue problem for $\frac{d^2}{dx^2}$ on $[0, L]$.

Dirichlet eigenvalue problem

- nontrivial solutions only when $\lambda = \left(\frac{n\pi}{L}\right)^2$
- corresponding solutions are $\sin\left(\frac{n\pi}{L}x\right)$

Definition

We call $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ an eigenvalue and $\sin\left(\frac{n\pi}{L}x\right)$ the corresponding eigenfunction.

Definition

The boundary-value problem

$$\begin{cases} y'' + \lambda y &= 0 \\ y'(0) &= 0 \\ y'(L) &= 0 \end{cases}$$

is called the Neumann eigenvalue problem for $\frac{d^2}{dx^2}$ on $[0, L]$.

Neumann eigenvalue problem

- nontrivial solutions only when $\lambda = \left(\frac{n\pi}{L}\right)^2$
- corresponding solutions are $\cos\left(\frac{n\pi}{L}x\right)$

Definition

We call $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ an eigenvalue and $\cos\left(\frac{n\pi}{L}x\right)$ the corresponding eigenfunction.

Definition

A boundary-value problem

$$\begin{cases} \frac{d}{dx}\left(r(x)y'\right) + \left(q(x) + \lambda p(x)\right)y & = 0 \\ A_1 y(a) + B_1 y'(a) & = 0 \\ A_2 y(b) + B_2 y'(b) & = 0 \end{cases}$$

is called a regular Sturm-Liouville problem on $[a, b]$. We assume that $r(x) > 0$, $p(x) > 0$, $\lambda \geq 0$, and at least one of A_1, B_1 and at least one of A_2, B_2 is nonzero.

Definition

If the regular Sturm-Liouville problem with $r, q, p, A_1, A_2, B_1, B_2, \lambda$ has a nontrivial solution, we call λ an eigenvalue and the solution an eigenfunction.

Dirichlet problem for $\frac{d^2}{dx^2}$

$$\begin{cases} \frac{d}{dx}\left(1y'\right) + \left(0 + \lambda 1\right) y & = 0 \\ 1y(a) + 0y'(a) & = 0 \\ 1y(b) + 0y'(b) & = 0 \end{cases}$$

Neumann problem for $\frac{d^2}{dx^2}$

$$\begin{cases} \frac{d}{dx}\left(1y'\right) + \left(0 + \lambda 1\right) y & = 0 \\ 0y(a) + 1y'(a) & = 0 \\ 0y(b) + 1y'(b) & = 0 \end{cases}$$

Theorem

For any regular Sturm-Liouville problem, we have:

1. *There is one eigenvalue for each natural number n, and if they are arranged in increasing order*

   $$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

   we have $\lambda_n \to \infty$ *as* $n \to \infty$
2. *Each eigenvalue has exactly one corresponding eigenfunction (up to scalar multiples).*
3. *Eigenfunctions for distinct eigenvalues are linearly independent.*
4. *The set of eigenfunctions is orthogonal with respect to the weight function* $p(x)$.
5. *The set of eigenfunctions is complete on* $[0, L]$.

symmetric $n \times n$ matrix	SLP
eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$	eigenvalues $\lambda_1 < \lambda_2 \cdots$
eigenvectors	eigenfunctions
eigenvectors are orthogonal	eigenfunctions orthogonal wrt p
eigenvectors form a basis	eigenfunctions are complete

Theorem

Assume $a(x) > 0$ *and* $d(x) > 0$ *on* $[a, b]$. *Then the general eigenvalue problem*

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0$$

has solutions for $\lambda_1 < \lambda_2 < \cdots \to \infty$.
Each eigenvalue has one eigenfunction (up to scalar multiples), and the eigenfunctions are a complete orthogonal set on $[a, b]$ *with respect to the weight function*

$$p(x) = \frac{d(x)}{a(x)} e^{\int \frac{b(x)}{a(x)} dx}$$

Definition

The differential equation

$$\frac{d}{dx}\left(r(x)y'\right) + \left(q(x) + \lambda p(x)\right) y = 0$$

along with one of the following conditions:

- $r(a) = 0$ and $Ay(b) + By'(b) = 0$
- $r(b) = 0$ and $Ay(a) + By'(a) = 0$

is called a singular Liouville problem on $[a, b]$.

Definition

The differential equation

$$\frac{d}{dx}\left(r(x)y'\right) + \left(q(x) + \lambda p(x)\right) y = 0$$

along with the condition $r(a) = r(b)$, $y(a) = y(b)$, $y'(a) = y'(b)$ is called a periodic Liouville problem on $[a, b]$.

Theorem

Suppose $\lambda \neq \mu$, and ϕ, ψ are corresponding solutions of a singular Liouville problem. Then ϕ and ψ are orthogonal with respect to $p(x)$.

Theorem

Suppose $\lambda \neq \mu$, and ϕ, ψ are corresponding solutions of a periodic Liouville problem. Then ϕ and ψ are orthogonal with respect to $p(x)$.

Note

The other properties of regular Sturm-Liouville problems may not apply to singular and periodic Sturm-Liouville problems!