Bessel and Legendre series
textbook section 12.6

MATH 241

January 26, 2012

Review: Legendre polynomials

Legendre's equation:

$$(1-x)^2 y'' - 2xy' + n(n+1)y = 0$$

Solutions are polynomials.

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Rodrigues' Formula

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \right]$$

Recurrence relation

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0$$

Legendre's equation has $r(x) = 1 - x^2$, so the problem is periodic and singular on the interval $[-1, 1]$.
The set of Legendre polynomials is complete and orthogonal on $[-1, 1]$.

Theorem

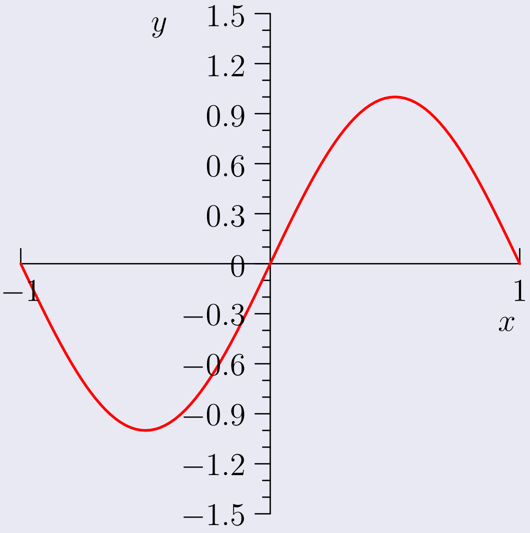
Any piecewise differentiable function $f(x)$ on $[-1, 1]$ with a continuous derivative has a Legendre series representation:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

Legendre Series for $sin(\pi x)$, $-1 \le x \le 1$



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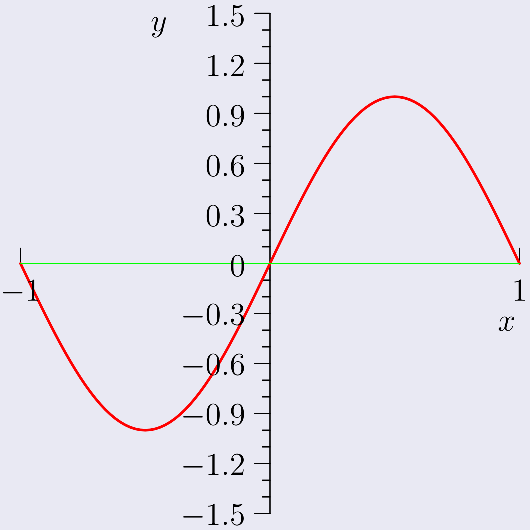

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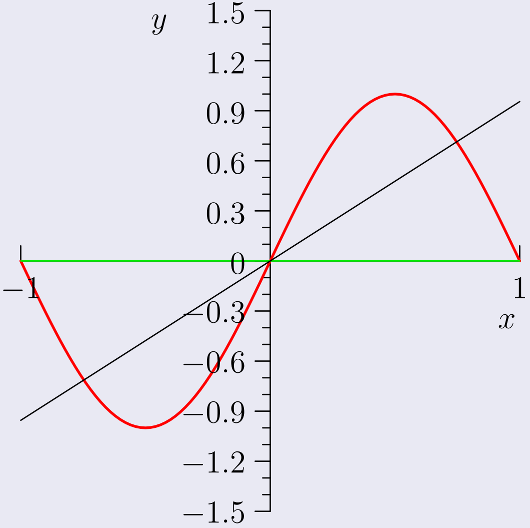

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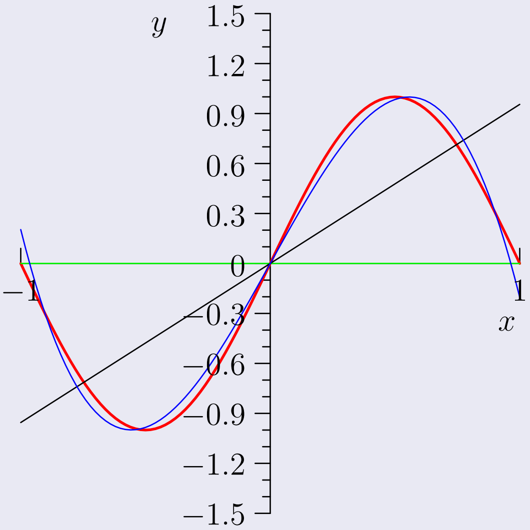

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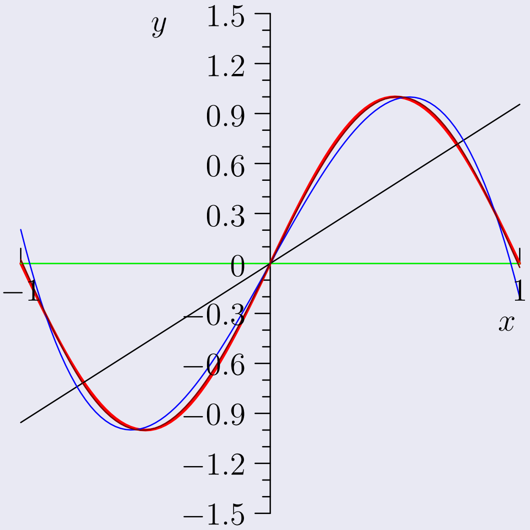
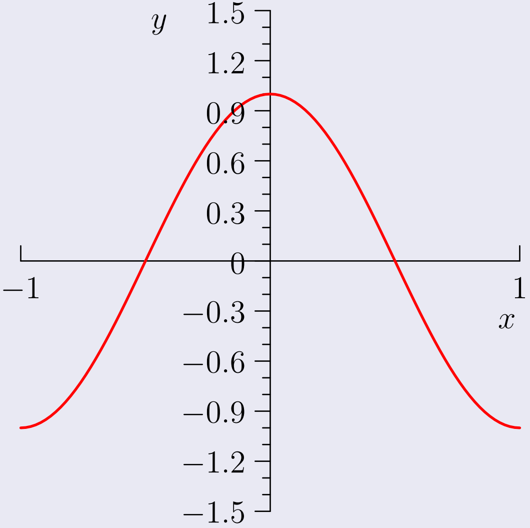
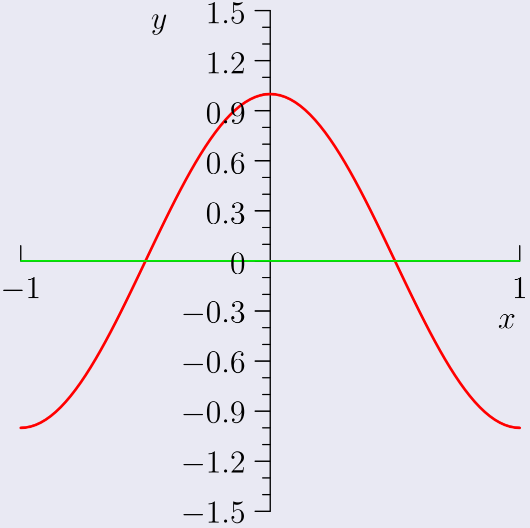


Legendre Series for $cos(\pi x), -1 \leq x \leq 1$



Legendre Series for $cos(\pi x)$, $-1 \le x \le 1$



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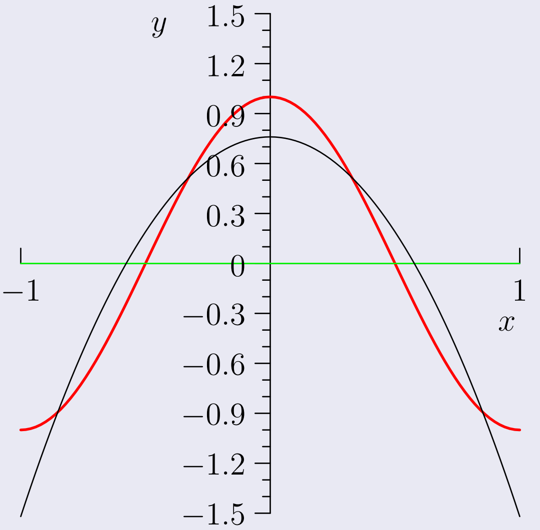
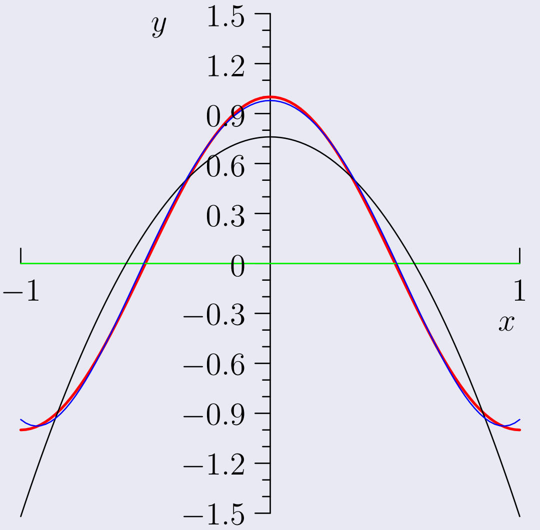


Legendre Series for $cos(\pi x)$, $-1 \leq x \leq 1$



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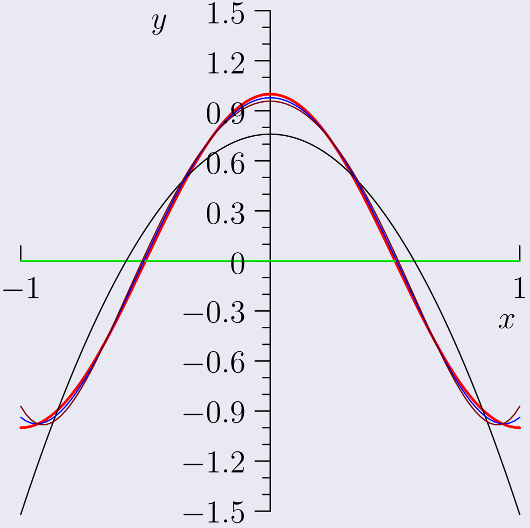


Theorem

Any piecewise differentiable function $f(x)$ on $[-1,1]$ with a continuous derivative has a Legendre series representation:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

The representation converges to $f(x)$ if f is continuous at x, and to $\frac{1}{2}\left(\lim_{t\to x^-} f(t) + \lim_{t\to x^+} f(t)\right)$ if f is not continuous at x.

Review: Bessel functions

Bessel's equation:

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

Two independent solutions, $J_n(x)$ ("Bessel function of the first kind") and $Y_n(x)$ ("Bessel function of the second kind").

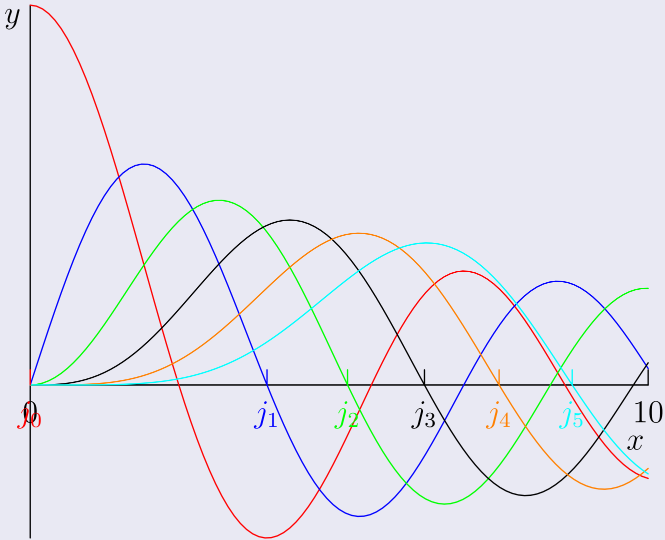
Recurrence relations

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

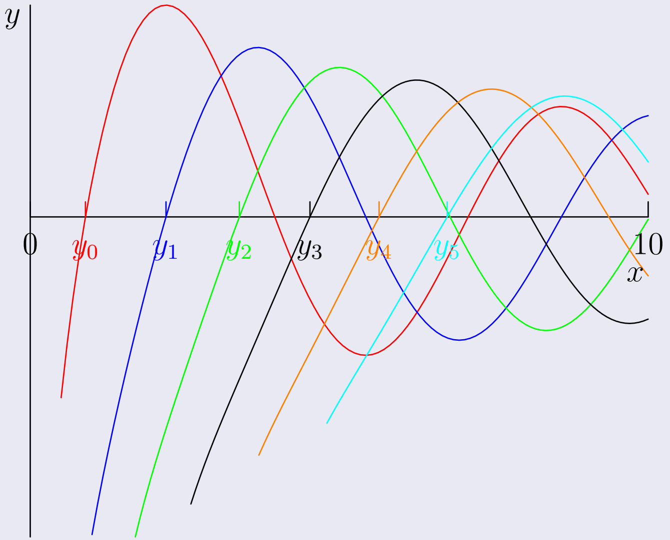
$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

Bessel functions of the first kind



Bessel functions of the second kind



Bessel basis

For a fixed n, the Bessel functions $\{J_n(\alpha_i x)\}$ are eigenfunctions for the Bessel singular Sturm-Liouville problem:

$$x^2y'' + xy' + (\alpha^2x^2 - n^2)y = 0$$
$$Ay(b) + By'(b) = 0$$

Since $p(x) = x$, the $\{J_n(\alpha_i x)\}$ are orthogonal with respect to $p(x) = x$:

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx = 0$$

Theorem

Any piecewise differentiable function $f(x)$ on $[0, b]$ with a continuous derivative has a Bessel series representation:

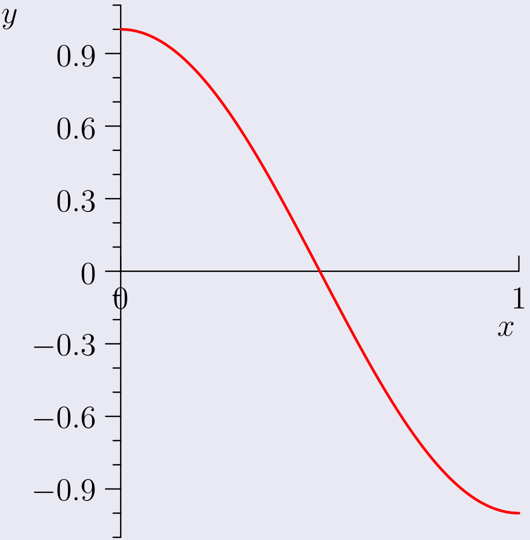
$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

where

$$c_i = \frac{1}{\|J_n(\alpha_i x)\|^2} \int_0^b x f(x) J_n(\alpha_i x) dx$$

The representation converges to $f(x)$ if f is continuous at x, and to $\frac{1}{2}\left(\lim_{t\to x^-} f(t) + \lim_{t\to x^+} f(t)\right)$ if f is not continuous at x.

Dirichlet $n = 1$ Bessel Series for $\cos(\pi x), 0 \leq x \leq 1$



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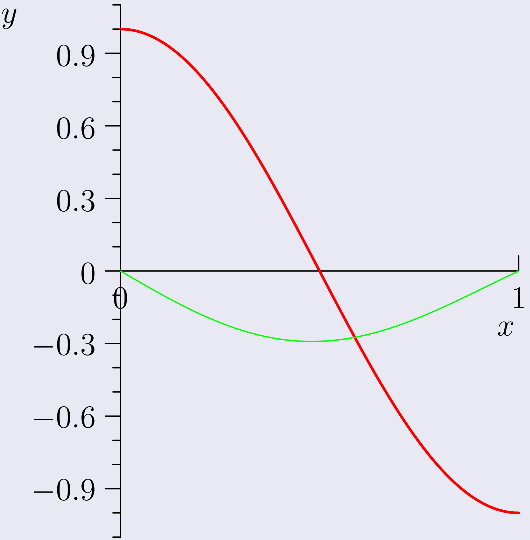

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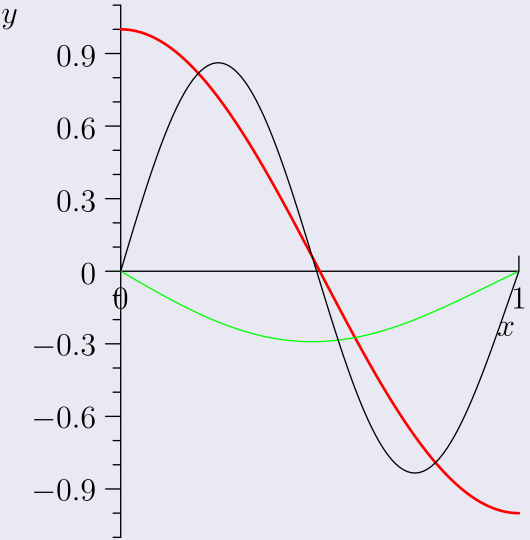

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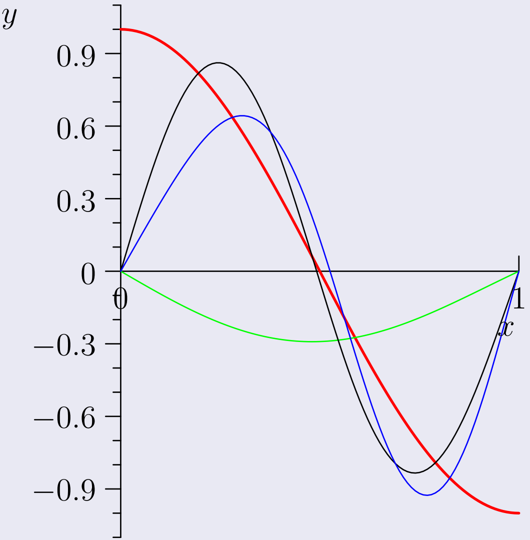

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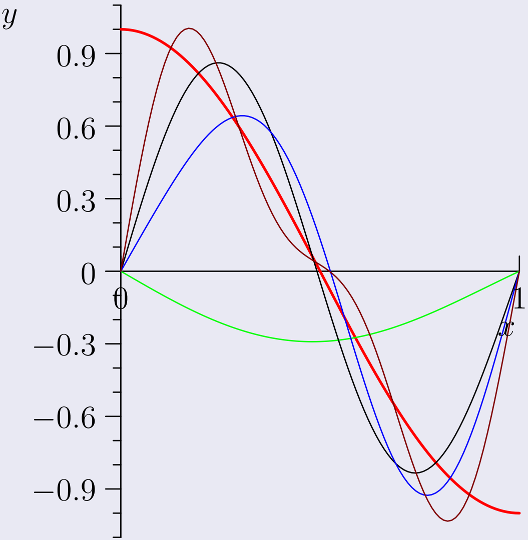


Bessel coefficients

The coefficients for the Dirichlet Bessel series are given by

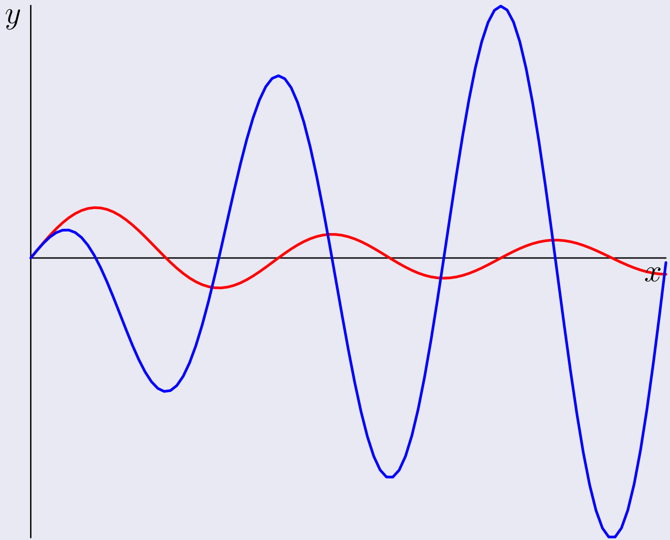
$$c_i = \frac{2}{b^2 J_{n+1}^2(j_{n,i})} \int_0^b x J_n(\frac{j_{n,i}}{b} x) f(x) dx$$

where $j_{n,i}$ is the i-th positive zero of J_n.
The coefficients for the Bessel series with boundary condition $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$ are given by

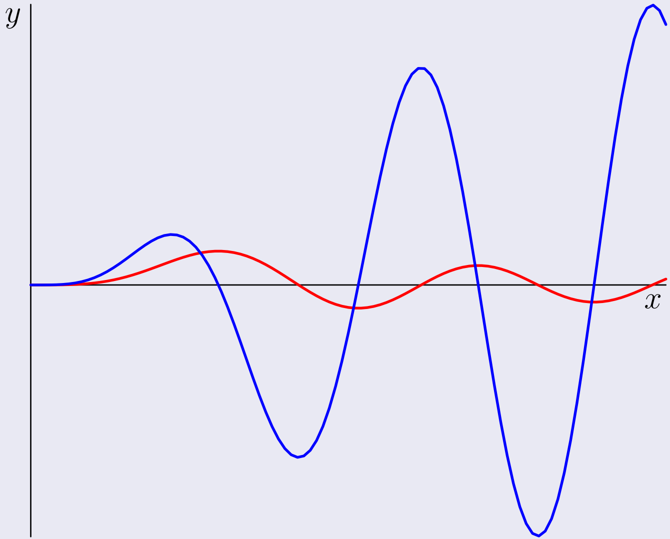
$$c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

where α_i is the i-th positive root of $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$.

Finding positive roots of $hJ_n(x) + xJ_n'(x) = 0$, $h = 1$, $n = 1$



Finding positive roots of $hJ_n(x) + xJ_n'(x) = 0$, $h = 1$, $n = 4$



Bessel coefficients, $n = 0$

The coefficients for the Bessel series with boundary condition $J_0'(\alpha b) = 0$ are given by

$$c_1 = \frac{2}{b^2} \int_0^b x f(x) dx$$

$$c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b x J_0(\alpha_i x) f(x) dx$$

where α_i is the i-th nonnegative root of $J_0'(\alpha b) = 0$.

Finding positive roots of $J_0'(x) = 0$