Separable PDE
textbook section 13.1

MATH 241

January 31, 2012

Definition

A (two-dimensional) linear second-order Partial Differential Equation is an equation of the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where A, B, C, D, E, F, and G are functions of x, y.

The equation is homogeneous if $G(x, y) = 0$, otherwise nonhomogeneous.

Homogeneous equations always have the trivial solution $u(x, y) = 0$.

Ordinarily (algebra, ODE), “solve” means “find all solutions of” or “find a general solution of”.

Solving PDE

. . . is just too hard. So we will have to settle for finding particular solutions, usually by imposing an ansatz or assumption.

(German an (“fore-”) + satz (“sentence”) = ansatz “hypothesis”)

Definition

A PDE is separable if it has a nontrivial solution of the form

$$u(x, y) = X(x)Y(y)$$

Separable PDE

If $u(x, y) = X(x)Y(y)$, then

$$\frac{\partial u}{\partial x} = X'Y$$

$$\frac{\partial^2 u}{\partial x^2} = X''Y$$

$$\frac{\partial u}{\partial y} = XY'$$

$$\frac{\partial^2 u}{\partial y^2} = XY''$$

$$\frac{\partial^2 u}{\partial x \partial y} = X'Y'$$

(Finite) Superposition Principle

If $u_1, \ldots, u_k$ are solutions to a homogeneous linear PDE, and $c_1, \ldots, c_k$ are real numbers, then their linear combination

$$\sum_{i=1}^{k} c_i u_i$$

is also a solution.

Transfinite Superposition Principle

If $u_1, u_2, \ldots$ are solutions to a homogeneous linear PDE, and $c_1, c_2, \ldots$ are real numbers, then their linear combination

$$\sum_{i=1}^{\infty} c_i u_i$$

is also a solution.

Definition

The homogeneous two-dimensional second order linear PDE

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

is called:

hyperbolic	if $B^2 - 4AC > 0$
parabolic	if $B^2 - 4AC = 0$
elliptic	if $B^2 - 4AC < 0$