The Heat Equation
textbook sections 13.2-13.3

MATH 241

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The Heat Equation

We want to describe how the temperature u in a thin rod varies over time.

assumption the rod is homogeneous

assumption the rod is perfectly insulated (except at the ends)

assumption temperature only depends on x and t

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

The heat equation is parabolic.

If we insulate both ends of the rod, then no heat can leave across the ends. So

$$\frac{\partial u}{\partial x}|_{x=0} = 0 \qquad\qquad \frac{\partial u}{\partial x}|_{x=L} = 0$$

This is the Neumann boundary condition.

If we cool both ends of the rod, then

$$u(0,t) = 0 \qquad\qquad u(L,t) = 0$$

This is the Dirichlet boundary condition.

We also need to specify $u(x,0)$, the initial data

Finding Separable Solutions of the Heat Equation

1. separate variables to get $T(t) = ce^{-\alpha^2 t}$, $X(x) = a\cos(\alpha x) + b\sin(\alpha x)$
2. use boundary conditions to find α (SLP!)
3. recognise that

$$\sum_{n=0}^{\infty} c_n e^{-\alpha_n^2 t} \left(a_n \cos(\alpha_n x) + b_n \sin(\alpha_n x)\right)$$

   formally solves the heat equation
4. $c_n a_n$ and $c_n b_n$ are the coefficients of the series representation of the initial data $u(x, 0)$