MATH 241 Practice Exam 1

This exam consists of 6 multiple-choice questions, 3 true-false questions, and one open-ended question. Show *all* your work. You will receive credit for a correct answer only if *your work is shown* and *your work supports your answer*. No credit will be given for correct answers which are not supported by work.

You will have 80 minutes to complete this exam.

$J_n(x)$ is the n-th Bessel function of the first kind; $j_{n,i}$ is the ith positive zero of J_n.

1. All solutions of the boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} & = u \\ u(0,t) & = 0 \\ u(\pi,t) & = 0 \end{cases}$$

which have the form $u(x,t) = X(x)T(t)$ are linear combinations of:

A. $e^{(n^2+1)t}\sin(nx),\ n > 0$ B. $e^{-(n+1)^2 t}\sin(nx),\ n > 0$

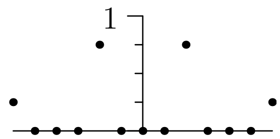
C. $\sin(nx)\sinh\left(\sqrt{n^2+1}t\right), \sin(nx)\cosh\left(\sqrt{n^2+1}t\right),\ n > 0$

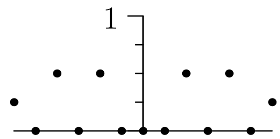
D. $\sin\left(\sqrt{n^2+1}t\right)\sin(nx), \cos\left(\sqrt{n^2+1}t\right)\sin(nx),\ n > 0$

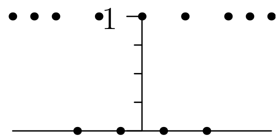
E. $e^{(1+n)^2 t}\cos(nx),\ n \geq 0$

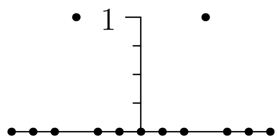
F. $\cos(nx)\sin\left(\sqrt{n^2+1}t\right), \sin(nx)\cos\left(\sqrt{n^2+1}t\right),\ n \geq 0$

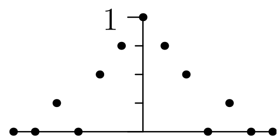
2. Let $f(x) = \sin^3(2x)$. Which of the following could be the frequency spectrum of f on $[-\pi, \pi]$?

A. 

B. 

C. 

D. 

E. 

3. Find the product of the first three eigenvalues of the Sturm-Liouville problem

$$\begin{cases} y'' + \lambda y & = 0 \\ y'(0) = y'(5) & = 0 \end{cases}$$

A. 1 B. $\frac{6\pi^3}{125}$ C. $\frac{2\pi^3}{125}$ D. $\frac{6\pi^2}{25}$ E. $\frac{2\pi^2}{25}$ F. 0

4. Expand $f(x) = x^4$ in the $n = 4$ Dirichlet Bessel series on $[0, 1]$.

A. $\sum_{i=1}^{\infty} \frac{2}{j_{4,i}^2 J_5(j_{4,i})^2} J_5(j_{4,i}x)$ B. $\sum_{i=1}^{\infty} \frac{2}{j_{5,i}^2 J_5(j_{5,i})} J_4(j_{5,i}x)$ C. $\sum_{i=1}^{\infty} \frac{2}{j_{4,i} J_5(j_{4,i})} J_4(j_{4,i}x)$

D. $\sum_{i=0}^{\infty} \frac{2}{J_5(j_{4,i})^2} J_4(j_{4,i}x)$ E. $\sum_{i=1}^{\infty} \frac{2}{j_{5,i} J_5(j_{5,i})} J_4(j_{4,i}x)$ F. $\sum_{i=1}^{\infty} \frac{2}{j_{4,i}^2 J_5(j_{4,i})} J_4(j_{4,i}x)$

5. Circle all of following which are true of the Fourier series on $[-2, 2]$ for

$$f(x) = \begin{cases} 1 + x^2 & 0 < x < 2 \\ 1 - x^2 & -2 < x < 0 \end{cases}$$

A. $a_0 = 2$ and the other a_n are all zero. B. a_n is negative if n is even

C. a_n is positive if n is odd D. the b_n are all zero.

E. b_n is negative if n is even F. b_n is positive if n is odd

6. Let c_n be the complex Fourier coefficient for $f(x) = 1 + x$ on $[-2, 2]$. Find the sum $c_0 + c_1 + c_2$.

A. $1 - \frac{2}{\pi} + \frac{1}{\pi}$ B. $1 - \frac{1}{\pi}i$ C. $-\frac{1}{2\pi}$ D. 0 E. $1 + \frac{3}{2\pi}i$ F. $-\frac{1}{2\pi}i$

7. TRUE or FALSE. For each of the following statements, indicate whether it is always true (T) or sometimes false (F). Support your answers.

   i The square of any function is an even function.

   ii Let $f(x)$ be a continuous function defined on $[0, L]$ such that $f(L) = 0$. If we reflect the function in an odd way, to $[-L, L]$ and compute its sine series

$$\sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right)$$

   then the series converges to $f(x)$ for all x in $[0, L]$.

iii The boundary value problem

$$\begin{cases} x^2y'' + 2xy' + x\lambda y &= 0 \\ y(1) = y(2) &= 0 \end{cases}$$

is a regular Sturm-Liouville problem.

8. A string of length 2π is vibrating according to:

$$\begin{aligned} u(0,t) = u(2\pi,t) &= 0 \\ u(x,0) &= \sin^2(x) \\ \frac{\partial u}{\partial t}(x,0) &= 0 \end{aligned}$$

Show that, for any x, $u(x,\pi) = 0$. Interpret this statement physically.