MATH 241 Practice Exam 2

This exam consists of 5 multiple-choice questions, 2 true-false questions, and 2 open-ended questions. Show *all* your work. You will receive credit for a correct answer only if *your work is shown* and *your work supports your answer*. No credit will be given for correct answers which are not supported by work.

You will have 80 minutes to complete this exam.

1. Heat is diffusing in a semi-infinite rod. One end of the rod is insulated. The initial temperature distribution is given by

$$u(x,0) = \begin{cases} 1 & 0 < x < 1 \\ e^{-x+1} & x \geq 1 \end{cases}$$

Find $u(x,t)$.

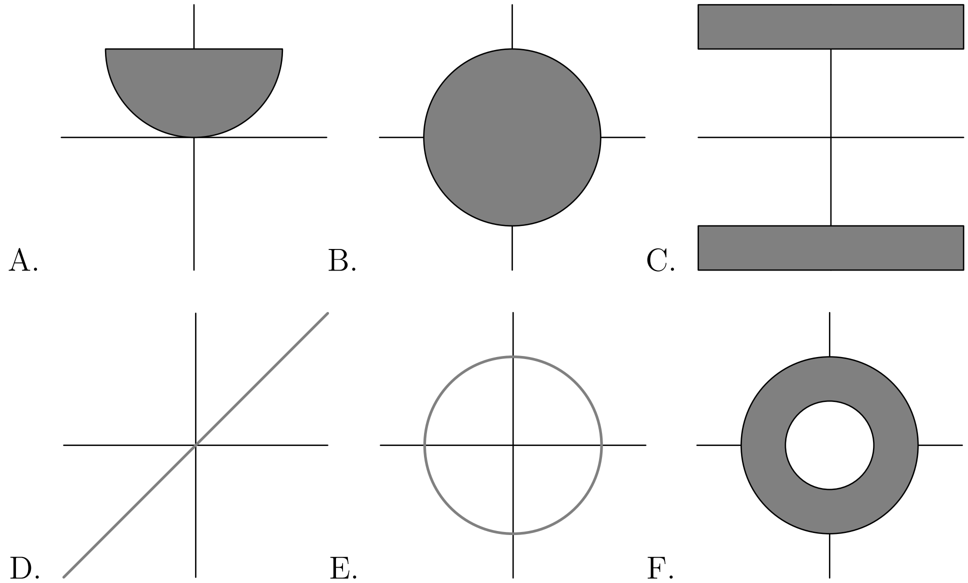
A. $u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha}{\alpha} e^{-\alpha^2 t}\sin(\alpha x)d\alpha$ B. $u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\alpha\sin\alpha - \cos\alpha}{1+\alpha^2} e^{-\alpha^2 t}\sin(\alpha x)d\alpha$

C. $u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\alpha}{1+\alpha^2}\sinh(\alpha t)\cos(\alpha x)d\alpha$ D. $u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\alpha}{1+\alpha^2} e^{-\alpha^2 t}\cos(\alpha x)d\alpha$

E. $u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha - \cos\alpha}{1+\alpha^2}\cosh(\alpha t)\sin(\alpha x)d\alpha$

F. $u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha - \alpha\sin\alpha}{1+\alpha^2}\right) e^{-\alpha^2 t}\cos(\alpha x)d\alpha$

2. Which of the following shaded sets are *regions*?



3. Solve the following Laplace's equation problem in the first quadrant:

$$\begin{cases} u_{xx} + u_{yy} = 0 & x > 0, y > 0 \\ u_x(0, y) = 0 & \\ u(x, 0) = \begin{cases} 0 & 0 < x < 1 \\ 1 & 1 \leq x < 2 \\ 0 & x \geq 2 \end{cases} & \end{cases}$$

Assume the solution has $u(x, y)$ bounded as $y \to \infty$.

A. $u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sin(2\alpha)}{\alpha} e^{-\alpha y} \sin(\alpha x) d\alpha$ B. $u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\alpha^2}{1+\alpha^2} e^{-\alpha y} \cos(\alpha x) d\alpha$

C. $u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sin(2\alpha) - \sin \alpha}{\alpha} e^{-\alpha y} \cos(\alpha x) d\alpha$

D. $u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\alpha \cos(2\alpha) - \sin \alpha}{\alpha} e^{\alpha y} \cos(\alpha x) d\alpha$

E. $u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha}{1+\alpha^2} \sinh(\alpha y) \sin(\alpha x) d\alpha$ F. $u(x, y) = \frac{2}{\pi} \int_0^\infty \alpha^2 \cosh(\alpha y) \cos(\alpha x) d\alpha$

4. A circular plate of radius 2 is at thermal equilibrium. Along its edge, the temperature is given by $\sin^2\theta + \sin\theta$. Find the temperature at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

A. 1 B. 0 C. $\frac{1}{2}$ D. $-\frac{1}{2}$ E. $\frac{1}{2} + \frac{\sqrt{2}}{4}$ F. $\sqrt{2}$

5. Circle all of the numbers z below with the property that two fifth roots of z lie in the first quadrant.

A. 1 B. -1 C. i D. $-i$ E. $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

6. An semi-infinite string vibrates according to

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} & x > 0, t > 0 \\ u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = e^{-x} \\ \frac{\partial u}{\partial x}(0,t) = 0 \end{cases}$$

Find $u(x,t)$.

7. TRUE or FALSE. For each of the following statements, indicate whether it is true (T) or false (F). Support your answers.

   i The principal n^{th} root of a complex number always lies in the first quadrant.

   ii The function $f(z) = \frac{\Re\mathfrak{e}(z)}{\Im\mathfrak{m}(z)}$ has a limit at $z = 0$.

8. In a short paragraph, explain the humour in the following voicemail message:

> Hello. You have reached an imaginary number. Please hang up, rotate your phone ninety degrees, and try calling back.