Name:

Math 104 Summer 2007: Quiz 3

There are 100 points available. Not all questions are necessarily the same difficulty, nor does difficulty necessarily increase along with the problem number. SHOW YOUR WORK - I'm very willing to give partial credit, but only for what I see, not for what I assume you were thinking. Good luck!

1. (20 points, 10 points each)

a. Estimate $\int_1^9 \frac{t+3}{t+2}dt$ using the trapezoidal rule with $n = 8$. You do not have to simplify.

We have $T_8 = \frac{1}{2\cdot 1}[f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + 2f(6) + 2f(7) + 2f(8) + f(9)] = \frac{1}{2}[4/3 + 2\cdot 5/4 + 2\cdot 6/5 + 2\cdot 7/6 + 2\cdot 8/7 + 2\cdot 9/8 + 2\cdot 10/9 + 2\cdot 11/10 + 12/11]$

b. Determine the exact value of the integral.
We have $\int_1^9 \frac{t+3}{t+2}dt = \int_1^9 \frac{t+2+1}{t+2}dt = \int_1^9 [1 + \frac{1}{t+2}]dt = t + \ln(t+2)|_1^9 = 9 + \ln(11) - [1 + \ln(3)] = 8 + \ln(11/3)$

2. (20 points, 10 points each) Evaluate the following integrals using the method of partial fractions:

a.) $\int \frac{4}{x^4-1}dx$

Firstly, $\frac{4}{x^4-1} = \frac{4}{(x^2-1)(x^2+1)} = \frac{4}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$.

Clearing the denominator of the last equation, we get:
$4 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$

Setting $x = 1$: $4 = 4A$, so $A = 1$.
Setting $x = -1$: $4 = -4B$, so $B = -1$.
Examining x^3 coefficients: $0 = A + B + C = 1 - 1 + C$, so $C = 0$. Examining x^0 coefficients: $4 = A - B - D = 1 + 1 - D$, so $D = -2$.

We now have: $\int \frac{4}{x^4-1}dx = \int (\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1})dx = \ln|x-1| - \ln|x+1| - 2\arctan(x) + C$

b.) $\int \frac{4x^2+3x+2}{x^3+x^2}dx$

Firstly, $\frac{4x^2+3x+2}{x^3+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$.

Clearing the denominator of the last equation, we get:
$4x^2 + 3x + 2 = Ax(x+1) + B(x+1) + Cx^2$

Setting $x = 0$: $2 = B$.
Setting $x = -1$: $4 - 3 + 2 = C$, so $C = 3$.
Setting $x = 1$: $9 = 2A + 2B + C = 2A + 4 + 3$, so $A = 1$.

We now have $\int \frac{4x^2+3x+2}{x^3+x^2}dx = \int (\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x+1})dx = \ln|x| - \frac{2}{x} + 3\ln|x+1| + C$.

3. (20 points, 10 points each) Evaluate each integral.

a. $\int \sec^2(x)\tan(x)\ln(\sec(x))dx$. (HINT: first make a substitution, then use a different integration technique from chapter 8.)

Setting $t = \sec(x)$, we have $dt = \sec(x)\tan(x)dx$, and our integral becomes $\int t\ln(t)dt$. We now use integration by parts with $u = \ln(t)$ and $dv = tdt$, so $du = \frac{dt}{t}$ and $v = \frac{1}{2}t^2$. We now have $\int t\ln(t)dt = \frac{1}{2}t^2\ln(t) - \int \frac{1}{2}tdt = \frac{1}{2}t^2\ln(t) - \frac{1}{4}t^2 + C$.

Replacing u with $\sec(x)$, we then get the answer:
$\int \sec^2(x)\tan(x)\ln(\sec(x))dx = \frac{1}{2}\sec^2(x)\ln(\sec(x)) - \frac{1}{4}\sec^2(x) + C$.

b. $\int \frac{dx}{x+\sqrt[3]{x}}$ (HINT: Let $u = \sqrt[3]{x}$).

Since du does not appear in the original integral, we solve for x in terms of u and then differentiate to get du. Thus with $u = \sqrt[3]{x}$, we have $u^3 = x$, so $3u^2du = dx$. Our original integral becomes:

$\int \frac{3u^2du}{u^3+u} = 3\int \frac{udu}{u^2+1}$

If we now let $t = u^2 + 1$, then $dt = 2udu$ and our integral becomes:

$\frac{3}{2}\int \frac{dt}{t} = \frac{3}{2}\ln|t| + C$.

Rewriting t in terms of u and then x, we get that our answer is:

$\frac{3}{2}\ln(u^2+1) + C = \frac{3}{2}\ln|x^{2/3}+1| + C$

4. (20 points, 10 points each) Determine the following values:

a. The length of the curve $y = \frac{1}{12}x^3 + \frac{1}{x}$ from the point $(1, \frac{13}{12})$ to the point $(2, \frac{7}{6})$

In terms of x, the arc length is $\int_1^2 \sqrt{1 + (dy/dx)^2} dx$

In our case, $1 + (dy/dx)^2 = 1 + (\frac{1}{4}x^2 - x^{-2})^2 = (\frac{1}{4}x^2 + x^{-2})^2$.
Thus our integral is $\int_1^2 (\frac{1}{4}x^2 + x^{-2})dx = \frac{1}{12}x^3 - \frac{1}{x}|_1^2 = (2/3 - 1/2) - (1/12 - 1) = 2/12 + 11/12 = 13/12$.

b. The surface area of the solid generated when the curve $x = y^2 + 1$ from (1,0) to $(3, \sqrt{2})$ is rotated around the x-axis.

Since we already have x in terms of y we will attempt to integrate in terms of y, in which case the surface area is:

$2\pi \int_0^{\sqrt{2}} y\sqrt{1 + (dx/dy)^2} dy = 2\pi \int_0^{\sqrt{2}} y\sqrt{1 + (2y)^2} dy = 2\pi \int_0^{\sqrt{2}} y\sqrt{1 + 4y^2} dy.$

Now while we could use a trig substitution, we can instead let $u = 1+4y^2$ and $du = 8ydy$, making our integral: $2\pi \int \frac{1}{8}u^{1/2}du = 2\pi(\frac{1}{8} \cdot \frac{2}{3}u^{3/2}) = \frac{\pi}{6}(1 + 4y^2)^{3/2}|_1^{\sqrt{2}} = \frac{\pi}{6}[(1 + 4 \cdot 2)^{3/2} - (1 + 4 \cdot 0)^{3/2}] = \frac{\pi}{6}(27 - 1) = \frac{13\pi}{3}$.

Alternatively, in terms of x, we have $y = \sqrt{x - 1}$, so $dy/dx = \frac{1}{2\sqrt{x-1}}$ and the surface area is:
$2\pi \int_1^3 \sqrt{x - 1}\sqrt{1 + \frac{1}{4(x-1)}} dx = 2\pi \int_1^3 \sqrt{x - 1 + 1/4} dx = 2\pi(\frac{2}{3}(x - 3/4)^{3/2}|_1^3) = \frac{4\pi}{3}[(3 - 3/4)^{3/2} - (1 - 3/4)^{3/2}] = \frac{4\pi}{3}[(9/4)^{3/2} - (1/4)^{3/2}] = \frac{4\pi}{3}[27/8 - 1/8] = \frac{13\pi}{3}$.

5. (20 points) Determine whether the following integrals converge or diverge. (HINT: Doing the integral in part c. is not easy, is likely impossible, and is definitely not in your best interest. Instead, try to use the information you attained from part b.) NOTE: A correct answer without any correct work to back it up will earn NO credit.

a. (6 points) $\int_1^2 \frac{2t}{t^2-1} dt$

Since the integrand is discontinuous at $t = 1$, we rewrite the integral as $\lim_{b\to 1^+} \int_b^2 \frac{2t}{t^2-1} dt$. In the integral we let $u = t^2 - 1$, and $du = 2tdt$ (though partial fractions would work as well.) We then have $\int \frac{du}{u} = \ln|u|$. Substituting back for t, we get:

$\lim_{b\to 1^+} \ln(t^2 - 1)|_b^2 = \lim_{b\to 1^+} \ln(2^2 - 1) - \ln(b^2 - 1)$. Now as $\lim_{b\to 1^+} \ln(b^2 - 1) = -\infty$, our integral diverges.

b. (6 points) $\int_5^\infty e^{-t} dt$

As our integral length is infinite, we rewrite it as $\lim_{b\to\infty} \int_5^b e^{-t} dt = \lim_{b\to\infty} -e^{-t}|_5^b =$ $\lim_{b\to\infty} -e^{-b} - (-e^{-5}) = e^{-5}$. Thus, our integral converges.

c. (8 points) $\int_5^\infty \frac{\sin^4(t)}{e^t+1} dt$

While we cannot integrate our integrand we note that it is always positive and for all t, $\frac{\sin^4(t)}{e^t+1} \leq \frac{1}{e^t+1} \leq \frac{1}{e^t} = e^{-t}$.

Now since $\int_5^\infty e^{-t} dt$ converges, by our comparison theorem, $\int_5^\infty \frac{\sin^4(t)}{e^t+1} dt$ converges as well.