Name:

Math 104 Summer 2007: Quiz 4

There are 100 points available. Not all questions are necessarily the same difficulty, nor does difficulty necessarily increase along with the problem number. SHOW YOUR WORK - I'm very willing to give partial credit, but only for what I see, not for what I assume you were thinking. Good luck!

1. (40 points) Consider the parametric equations $x = t - 1$, $y = (t-1)^3$, with $1 \leq t \leq 4$.

a. (10 points) Sketch the graph. Be sure to indicate, with an arrow, the direction in which the curve is traced.

We first plot the points: $\begin{array}{ccccccc} t & = & 1 & 2 & 3 & 4 \\ x & = & 0 & 1 & 2 & 3 \\ y & = & 0 & 1 & 8 & 27 \end{array}$

and then sketch the graph, using these points as guides, as well as the fact that our graph traces out a portion of $y = x^3$. Also, our arrows point up and to the right.

b. (5 points) Set up (but do NOT evaluate) and integral describing the arc length of this curve.

We find $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 3(t-1)^2$. Thus our integral is $L = \int_1^4 \sqrt{1 + 9(t-1)^4}dt$

c. (15 points) Let S denote the surface obtained by rotating this curve about the x-axis. Set up (and DO evaluate) an integral describing the surface area of S.

The distance to the x-axis is $y = (t-1)^3$. Thus we have:

$S = \int_1^4 2\pi(t-1)^3\sqrt{1 + 9(t-1)^4}dt$

We now let $u = 1 + 9(t-1)^4$, so $du = 36(t-1)^3dt \Rightarrow \frac{du}{36} = (t-1)^3dt$. Also, in terms of u, the bounds of our integral are $1 + 9(1-1)^4 = 1$ and $1 + 9(4-1)^4 = 730$. Thus we now have:

$S = \int_1^{730} \frac{\pi}{18}u^{1/2}du = \frac{\pi}{18}[\frac{2}{3}u^{3/2}|_1^{730}] = \frac{\pi}{27}(730^{3/2} - 1)$

d. (10 points) What is $\frac{dy}{dx}$ when $t = 2$?

We have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(t-1)^2}{1}$
Thus, when $t = 2$, $\frac{dy}{dx} = 3$.

2. (20 points) The manager for a fast-food restaurant determines that the average time that her customers wait for service is 4 minutes. Using an exponentially decreasing probability density function (parts a. and b.), determine:

a.) (5 points) The probability that a customer has to wait for more than 5 minutes.

We'll refer to our exponentially decreasing probability density function with mean 4 as f. Then for $x \geq 0$, $f(x) = \frac{1}{4}e^{-x/4}$ and otherwise $f(x) = 0$.

Now $P(X > 5) = \int_5^\infty \frac{1}{4}e^{-x/4}dx = \lim_{t\to\infty}[-e^{-x/4}|_5^t] = e^{-5/4}$

b.) (5 points) The median waiting time. (That is, the waiting time m such that half the customers wait more than m minutes and half the customers wait less than m minutes.)

We want m such that $\frac{1}{2} = P(X < m) = \int_0^m \frac{1}{4}e^{-x/4}dx$

The integral becomes $-e^{x/4}|_0^m = 1 - e^{-m/4}$

We then have the chain of equalities:

$$\begin{aligned} 1 - e^{-m/4} &= 1/2 \\ e^{-m/4} &= 1/2 \\ e^{m/4} &= 2 \\ m/4 &= \ln(2) \\ m &= 4\ln(2) \end{aligned}$$

Thus the median waiting time is $4\ln(2)$ minutes.

c.) (10 points) The manager's new boss is convinced that no customer will ever wait more than 20 minutes and that the correct probability density function will be of the form:

$$f(x) = \begin{pmatrix} Ax\sqrt{400 - x^2} & 0 \leq x \leq 20 \\ 0 & \textit{otherwise} \end{pmatrix}.$$

What must be the value of A so that $f(x)$ is a probability density function?

We seek an $A \geq 0$ such that $\int_0^{20} Ax\sqrt{400 - x^2}dx = 1$.
To evaluate the integral, we let $u = 400 - x^2$, so $du = -2xdx$. Thus $xdx = -\frac{1}{2}du$ and the bounds in terms of u are $400 - 20^2 = 0$ and $400 - 0^2 = 400$. The integral is then:

$\int_{400}^0 -\frac{A}{2}u^{1/2}du = -\frac{A}{3}u^{3/2}|_{400}^0 = \frac{A}{3} \cdot 400^{3/2} = \frac{8000A}{3}$. So we now want A such that $\frac{8000A}{3} = 1$. Thus $A = \frac{3}{8000}$.

3. (20 points) Consider the four-leafed rose $r = \sin(2\theta)$.

a. (5 points) Set up (but do NOT evaluate) an integral giving the area enclosed by one of the leaves.

We first seek consecutive θ values for which $r = 0$. Two such values are $\theta = 0$ and $\pi/2$. Thus the desired integral is: $\int_0^{\pi/2} \frac{1}{2}[\sin(2\theta)]^2 d\theta$

b. (5 points) Set up (but do NOT evaluate) an integral giving the arclength of the entire curve.

We first calculate $\frac{dr}{d\theta} = 2\cos(2\theta)$. Now we can either multiply the length of one leaf by four or find the length of the entire curve at once. Thus we have either:

$L = 4\int_0^{\pi/2} \sqrt{\sin^2(2\theta) + 4\cos^2(2\theta)} d\theta$ OR $\int_0^{2\pi} \sqrt{\sin^2(2\theta) + 4\cos^2(2\theta)} d\theta$

c. (10 points) Use the equation $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ to write the similar (BUT DIFFERENT) curve $r^2 = \sin(2\theta)$ as a cartesian equation, i.e., find an equation $F(x, y) = 0$ that traces out the same curve.

We first have $r^2 = 2\sin(\theta)\cos(\theta)$. Multiplying each side of the equation by r^2, we get $r^4 = 2r^2\sin(\theta)\cos(\theta)$. Since we know that $r^2 = x^2 + y^2$, $y = r\sin(\theta)$ and $x = r\cos(\theta)$, we write the equation as $(r^2)^2 = 2(r\sin(\theta))(r\cos(\theta))$, so it is equivalent to $(x^2 + y^2)^2 = 2yx$. While this is fine as an answer, we can rewrite the equation as $(x^2 + y^2)^2 - 2yx = 0$ to conform exactly with the form $F(x, y) = 0$.

4. (20 points, 5 pts each) For each of the following sequences, determine whether they converge or diverge, and if they converge, determine the limit. NOTE: A correct answer without any correct work to back it up will earn NO credit.

a. $a_n = \cos(n)$

As $n \to \infty$, $\cos(n)$ oscillates among values between -1 and 1, not honing in on any particular number. Thus, the sequence diverges.

b. $a_n = \frac{n}{2n+1} + \cos(1/n)$

First, $\frac{n}{2n+1} = \frac{1}{2+1/n}$ which $\to \frac{1}{2}$ as $n \to \infty$
Also, $\frac{1}{n} \to 0$, so $\cos(\frac{1}{n}) \to \cos(0) = 1$. Thus a_n converges to $\frac{1}{2} + 1 = \frac{3}{2}$

c. $a_n = \frac{2\ln(n)}{n}$

Since the numerator and denominator go to ∞, we use L'Hopital's Rule. Thus $\lim_{n\to\infty} \frac{2\ln(n)}{n} = \lim_{n\to\infty} \frac{2/n}{1} = 0$

Thus, a_n converges to 0.

d. $a_n = \ln(\frac{2n+\ln(n)}{n})$

We first determine the limit of the quantity within the outermost parenthesis. Since the numerator and denominator go to ∞, we use L'Hopital's Rule. Thus $\lim_{n\to\infty} \frac{2n+\ln(n)}{n} = \lim_{n\to\infty} \frac{2+1/n}{1} = 2$.

Hence a_n converges to $\ln(2)$.