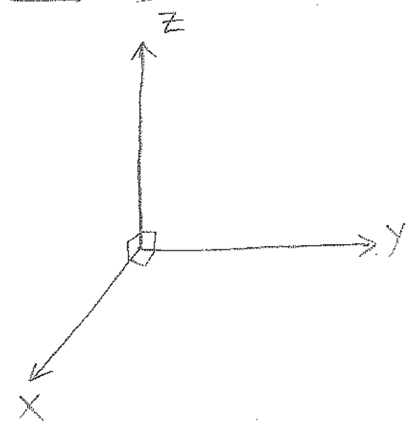
MATH 114

Lecture 1   Monday, 5/23

12.1 Three dimensional coordinate systems



- To describe points in space we use a right-handed coordinate frame with 3 coordinate axes: x, y and z.

- Any point in space is specified if we know its x, y and z coordinates:

$$P = (x, y, z)$$

- We have three coordinate planes
  1. the x-y plane : all points whose z-coordinate is zero i.e. points that look like $(x, y, 0)$
  2. the y-z plane : points that look like $(0, y, z)$
  3. the x-z plane : points that look like $(x, 0, z)$

- they can be described by the equations $z=0$, $x=0$, $y=0$ resp.

- Planes parallel to the coordinate planes

  e.g. $x=2$, $y=7$, $z=-1$

- Equations and inequalities describe sets (collections of points)

  e.g. 1. $x \geq 4$

  2. $x \geq 0, y \geq 0, z \geq 0$ ... this is called the first octant.

- Distance between points in 3-D space

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are two points, then

$$\underbrace{|P_1 P_2|} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

This is the notation for distance between P_1 and P_2.

e.g. What is the distance b/w $A = (1, 2, 3)$ and $B = (1, 5, 0)$?

$$|AB| = \sqrt{(1-1)^2 + (2-5)^2 + (3-0)^2}$$

$$= \sqrt{0^2 + (-3)^2 + 3^2}$$

$$= \sqrt{0 + 9 + 9}$$

$$= \sqrt{18}$$

NOTE : This formula is very similar to the one in 2-D !

- Equation for a sphere

A sphere with center (x_1, y_1, z_1) and radius a :

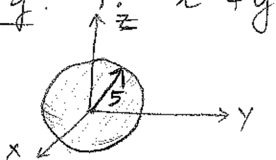
$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = a^2$$

NOTE 1: Notice the similarity to the equation for a circle in the plane

NOTE 2 : In words, this formula is saying:
"Consider all points (x, y, z) whose distance from the fixed point (x_1, y_1, z_1) is equal to a".

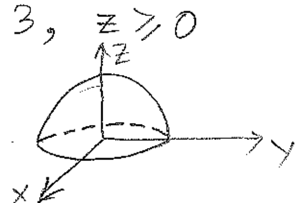
- Inequalities

e.g. 1. $x^2 + y^2 + z^2 \leq 5$

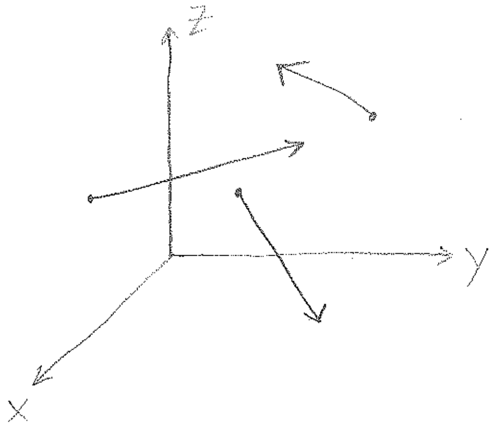


2. $x^2 + y^2 + z^2 = 3$, $z \geq 0$

(upper hemisphere)



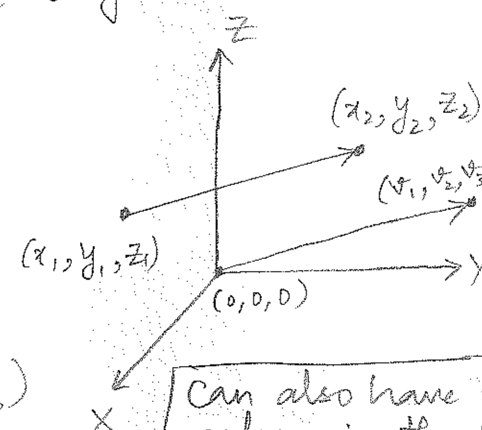
12.2 Vectors



- A vector is a directed line segment, i.e. it is an arrow with a starting point and an endpoint and some direction.
- Vectors are important in physics, where you have quantities like force and velocity, where both magnitude and direction are important.

- Notation: $\overrightarrow{AB}$, for a vector with starting point A and end point B.

  the magnitude (length) of $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$
- Two vectors are equal if they have same length and direction.



- Vector in standard position

  The vector with same length and direction, but starting point the origin $(0,0,0)$

  write $\vec{v} = \langle v_1, v_2, v_3 \rangle$ where (v_1, v_2, v_3) are coordinates of the endpoint when $\vec{v}$ is in standard position. $\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

  Can also have vectors in the plane, $\vec{v} = \langle v_1, v_2 \rangle$

  v_1, v_2, v_3 are called components of the vector, in the x, y, z directions.

- Length or magnitude: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- Zero vector: $\langle 0, 0, 0 \rangle$ has length 0 and no specific direction.

e.g. $A = (1, 2, -3)$, $B = (-5, 1, -2)$

$$\vec{v} = \overrightarrow{AB} = \langle -5-1, 1-2, -2-(-3) \rangle = \langle -6, -1, 1 \rangle$$

and $|\vec{v}| = \sqrt{(-6)^2 + (-1)^2 + 1^2} = \sqrt{36+1+1} = \sqrt{38}$

• Operations on vectors

1. Scalar multiplication: $\vec{u} = \langle u_1, u_2, u_3 \rangle$

$$k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$$

e.g. if $\vec{u} = \langle 3, -1, 4 \rangle$, then $3\vec{u} = \langle 9, -3, 12 \rangle$

if $\vec{v} = \langle 2, 4, 6 \rangle$, then $\frac{1}{2}\vec{v} = \langle 1, 2, 3 \rangle$

2. Addition $\vec{u} = \langle u_1, u_2, u_3 \rangle$; $\vec{v} = \langle v_1, v_2, v_3 \rangle$

then $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

the difference of two vectors:

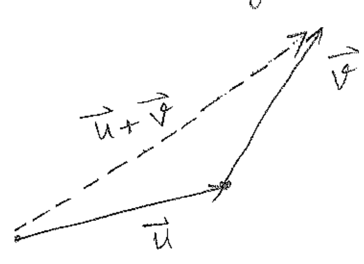
$$\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$

e.g. let $\vec{u} = \langle -1, 5, 2 \rangle$, $\vec{v} = \langle 6, 0, -3 \rangle$

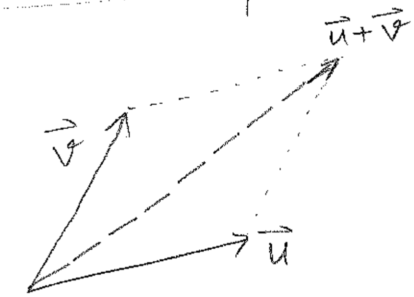
1. $\vec{u} - \vec{v} = \langle -1-6, 5-0, 2-(-3) \rangle = \langle -7, 5, 5 \rangle$

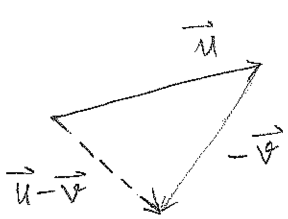
2. $2\vec{u} + \frac{1}{3}\vec{v} = \langle -2, 10, 4 \rangle + \langle 2, 0, -1 \rangle = \langle 0, 10, 3 \rangle$

• Geometrically:



Parallelogram law of vector addition





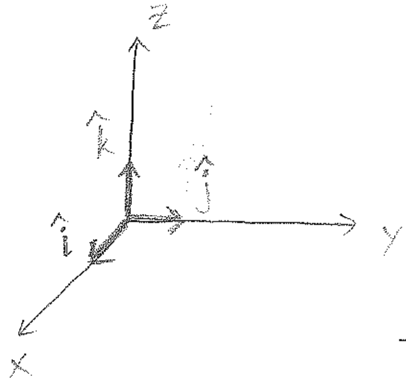
Properties of vector addition

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $\vec{u} + \vec{0} = \vec{u}$
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
4. $\vec{u} + (-\vec{u}) = 0$
5. $0\vec{u} = \vec{0}$
6. $a(b\vec{u}) = (ab)\vec{u}$
7. $(a+b)\vec{u} = a\vec{u} + b\vec{u}$
8. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

- Unit vectors

A unit vector is a vector of length 1.

Standard unit vectors : $\hat{i} = \langle 1,0,0\rangle$, $\hat{j} = \langle 0,1,0\rangle$, $\hat{k} = \langle 0,0,1\rangle$ (unit vectors in the directions of the coordinate axes)



Any vector $\vec{v} = \langle v_1, v_2, v_3\rangle$ can be written as a linear combination of the standard unit vectors:

$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$.

$\boxed{\dfrac{\vec{v}}{|\vec{v}|} \text{ is a unit vector in the direction of } \vec{v}}$

e.g. $\vec{u} = \langle 2,-1,-2\rangle$

$$|\vec{u}| = \sqrt{2^2+(-1)^2+(-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$\therefore$ the unit vector in direction of $\vec{u}$ is :

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\langle 2,-1,-2\rangle}{3} = \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}\right\rangle$$

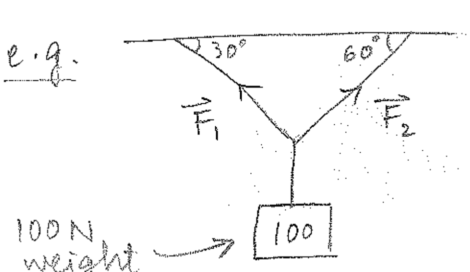
$$= \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

$\vec{u}$ can be written as $\vec{u} = |\vec{u}| \cdot \hat{u}$ = "length times direction"

$$\vec{u} = 3\left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

- Midpoint of a line segment joining $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is the point $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}, \dfrac{z_1+z_2}{2}\right)$

e.g.



Find $\vec{F_1}$ and $\vec{F_2}$.

Solution: The resultant force must balance the weight of the object.

$\vec{F_1} = \langle -|\vec{F_1}|\cos 30°, |\vec{F_1}|\sin 30°\rangle$; $\vec{F_2} = \langle |\vec{F_2}|\cos 60°, |\vec{F_2}|\sin 60°\rangle$

Force due to weight: $\vec{F} = \langle 0, -100 \rangle$

$\therefore \vec{F_1} + \vec{F_2} = -\vec{F} = \langle 0, 100 \rangle$

Therefore $-|\vec{F_1}|\cos 30^\circ + |\vec{F_2}|\cos 60^\circ = 0$

$|\vec{F_1}|\sin 30^\circ + |\vec{F_2}|\sin 60^\circ = 100$

i.e. $-|\vec{F_1}|\frac{\sqrt{3}}{2} + |\vec{F_2}|\frac{1}{2} = 0$, so $|\vec{F_2}| = \sqrt{3}|\vec{F_1}|$

$|\vec{F_1}| \cdot \frac{1}{2} + |\vec{F_2}| \cdot \frac{\sqrt{3}}{2} = 100$

i.e. $\frac{|\vec{F_1}|}{2} + \frac{3}{2}|\vec{F_1}| = 100$, i.e. $2|\vec{F_1}| = 100$

so $|\vec{F_1}| = 50$

and $|\vec{F_2}| = \sqrt{3} \times 50 = 1.73 \times 50 = 86.5$

$\vec{F_1} = \langle -50 \times \frac{\sqrt{3}}{2}, 50 \times \frac{1}{2} \rangle = \langle -25\sqrt{3}, 25 \rangle$

$\vec{F_2} = \langle \sqrt{3} \times 50 \times \frac{1}{2}, 50\sqrt{3} \times \frac{\sqrt{3}}{2} \rangle = \langle 25\sqrt{3}, 75 \rangle$

12.3 Dot Product

A way to take two vectors and get a number...

$\vec{u} = \langle u_1, u_2, u_3 \rangle$; $\vec{v} = \langle v_1, v_2, v_3 \rangle$

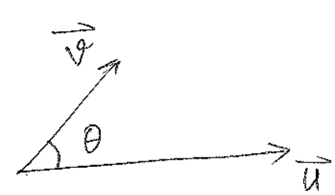
then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ "the dot product of $\vec{u}$ and $\vec{v}$"

e.g. $\langle 1, 2, -3 \rangle \cdot \langle 5, -7, 1 \rangle = 1 \times 5 + 2 \times (-7) + (-3) \times 1$

$= 5 - 14 - 3 = -12$

e.g. $(4\hat{i} + 2\hat{j} + 5\hat{k}) \cdot (-\hat{i} - \frac{1}{2}\hat{j} + 3\hat{k}) = -4 - 1 + 15 = 10$

Another formula: $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$



Orthogonal vectors : $\vec{u}$, $\vec{v}$ so that $\vec{u}\cdot\vec{v}=0$

this happens when either $\vec{u}$ or $\vec{v}$ is $\vec{0}$ $\boxed{\text{or}}$ when $\vec{u}$ and $\vec{v}$ are perpendicular $(\theta=\frac{\pi}{2})$

Formula for angle in terms of dot product :

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

e.g. $\vec{u}=\langle 1,5,0\rangle$, $\vec{v}=\langle -10,2,3\rangle$; $\theta=?$

$$\vec{u}\cdot\vec{v} = -10+10+0=0$$

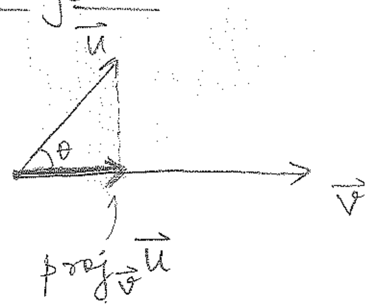
$$|\vec{u}|=\sqrt{1+25+0}=\sqrt{26}\ ,\quad |\vec{v}|=\sqrt{100+4+9}=\sqrt{113}$$

$$\theta = \cos^{-1}\left(\frac{0}{\sqrt{26}\cdot\sqrt{113}}\right) = \cos^{-1}(0)=\frac{\pi}{2}.$$

Properties of dot product

1. $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$
2. $(c\vec{u})\cdot\vec{v}=\vec{u}\cdot(c\vec{v})=c(\vec{u}\cdot\vec{v})$
3. $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$
4. $\vec{u}\cdot\vec{u}=|\vec{u}|^2$
5. $\vec{0}\cdot\vec{u}=0$

Projection



Projection of $\vec{u}$ onto $\vec{v}$.

$$\text{proj}_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta\cdot\frac{\vec{v}}{|\vec{v}|}$$

$$=\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\cdot\vec{v}$$

component of $\vec{u}$ in direction of $\vec{v}$ $=|\vec{u}|\cos\theta=\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$

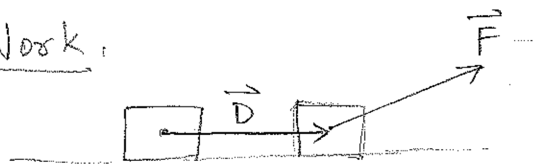
e.g. $\vec{u}=\frac{1}{2}\hat{i}+3\hat{j}+5\hat{k}$, $\vec{v}=-2\hat{i}+\hat{j}+2\hat{k}$

$$\text{proj}_{\vec{v}}\vec{u}=\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\ ;\quad \vec{u}\cdot\vec{v}=-1+3+10=12$$

$$|\vec{v}|=3$$

$$=\left(\frac{12}{9}\right)\cdot(-2\hat{i}+\hat{j}+2\hat{k}) = -\frac{8}{3}\hat{i}+\frac{4}{3}\hat{j}+\frac{8}{3}\hat{k}$$

Work:



Work done $= \vec{F} \cdot \vec{D}$

"force dot displacement"