MATH 114

$\boxed{\text{Lecture 2}}$ Tuesday, 5/24

Some stuff we couldn't get to last time:

1. If $\vec{v}$ is a given vector, then the unit vector in the direction of $\vec{v}$ is $\dfrac{\vec{v}}{|\vec{v}|}$

[because if $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

and $\dfrac{\vec{v}}{|\vec{v}|} = \left\langle \dfrac{v_1}{\sqrt{\cdots}}, \dfrac{v_2}{\sqrt{\cdots}}, \dfrac{v_3}{\sqrt{\cdots}} \right\rangle$

$$\text{so } \left|\frac{\vec{v}}{|\vec{v}|}\right| = \sqrt{\left(\frac{v_1}{\sqrt{\cdots}}\right)^2 + \left(\frac{v_2}{\sqrt{\cdots}}\right)^2 + \left(\frac{v_3}{\sqrt{\cdots}}\right)^2}$$

$$= \sqrt{\frac{v_1^2 + v_2^2 + v_3^2}{v_1^2 + v_2^2 + v_3^2}} = \sqrt{1} = 1$$

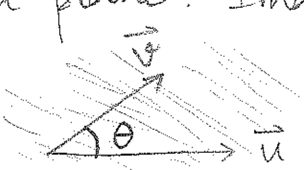
so $\dfrac{\vec{v}}{|\vec{v}|}$ is a unit vector.]

2. Midpoint of a line segment joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}, \dfrac{z_1+z_2}{2}\right)$.

We can see why this makes sense: we're doing the average of each coordinate.

12.4 The Cross Product

- If $\vec{u}$, $\vec{v}$ are vectors that are not parallel, they determine a plane. Imagine this...



Let $\hat{n}$ be the unit vector perpendicular to that plane with direction given by the right hand rule.

Right hand rule: When the fingers of your right hand curl from $\vec{u}$ to $\vec{v}$, the direction that the thumb points in, gives the direction of $\hat{n}$.

- DEFN: The cross product of $\vec{u}$ with $\vec{v}$ ("$\vec{u}$ cross $\vec{v}$") is the vector $\vec{u} \times \vec{v} = (|\vec{u}|\,|\vec{v}| \sin\theta)\,\hat{n}$

If $\vec{u}$ and $\vec{v}$ are parallel (point in same or opposite direction) define $\vec{u} \times \vec{v} = 0$.

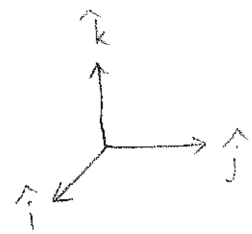
- Properties of the cross product:

$\vec{u}, \vec{v}, \vec{w}$ are vectors and r, s scalars then

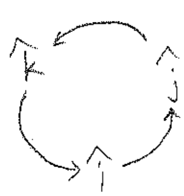
1. $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$
2. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3. $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$
4. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$
5. $\vec{0} \times \vec{u} = \vec{0}$
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

- Important: $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$

The direction of the normal will be opposite to that for $\vec{u} \times \vec{v}$ (using the right hand rule).



- Cross products of the standard unit vectors:



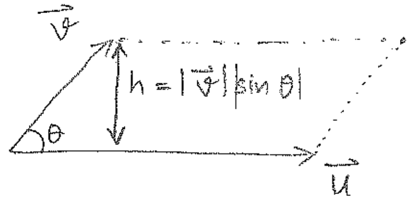
$\hat{i} \times \hat{j} = \hat{k} = -(\hat{j} \times \hat{i})$

$\hat{j} \times \hat{k} = \hat{i} = -(\hat{k} \times \hat{j})$

$\hat{k} \times \hat{i} = \hat{j} = -(\hat{i} \times \hat{k})$

and $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$.

- Area of a parallelogram



The area of the parallelogram having $\vec{u}$, $\vec{v}$ as adjacent sides, is:

$$\text{Area} = \text{base} \times \text{height}$$
$$= |\vec{u}| \cdot |\vec{v}||\sin\theta|$$

On the other hand, $|\vec{u} \times \vec{v}| = \left| \, |\vec{u}||\vec{v}| \sin\theta \, \hat{n} \right|$

$$= |\vec{u}||\vec{v}| \, |\sin\theta| \, \underbrace{|\hat{n}|}_{=1}$$

$$= |\vec{u}| \, |\vec{v}| |\sin\theta|$$

Upshot: Area of parallelogram $= |\vec{u} \times \vec{v}|$
(the magnitude of $\vec{u} \times \vec{v}$.)

- For computations, we use the determinant formula for $\vec{u} \times \vec{v}$:

$\vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$, $\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$

then $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

Examples (from the textbook)

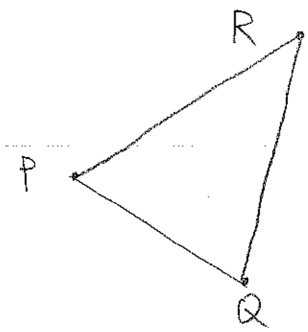
1. Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ if $\vec{u} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{v} = -4\hat{i} + 3\hat{j} + \hat{k}$

Solution: $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \hat{i}\,(1\times 1 - 3\times 1) - \hat{j}\,(2\times 1 - (-4)\times 1) + \hat{k}\,(2\times 3 - (-4)\times 1)$

$$= -2\hat{i} - 6\hat{j} + 10\hat{k}$$

$$\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v}) = 2\hat{i} + 6\hat{j} - 10\hat{k}$$

2. Find a unit vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$



Solution: $\overrightarrow{PQ} \times \overrightarrow{PR}$ is a vector perpendicular to the plane of P, Q and R.

$$\overrightarrow{PQ} = \langle 2-1,\ 1-(-1),\ -1-0 \rangle$$
$$= \langle 1, 2, -1 \rangle = \hat{i} + 2\hat{j} - \hat{k}$$

$$\overrightarrow{PR} = \langle -1-1,\ 1-(-1),\ 2-0 \rangle$$
$$= \langle -2, 2, 2 \rangle = -2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \hat{i}\left(2\times 2 - 2\times(-1)\right) - \hat{j}\left(1\times 2 - (-2)\times(-1)\right) + \hat{k}\left(1\times 2 - (-2)\times 2\right)$$

$$= 6\hat{i} + 6\hat{k}$$

What's the unit vector in this direction?

$$\frac{6\hat{i}+6\hat{k}}{|6\hat{i}+6\hat{k}|} = \frac{6\hat{i}+6\hat{k}}{\sqrt{6^2+6^2}} = \frac{6\hat{i}+6\hat{k}}{6\sqrt{2}} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}$$

- Triple scalar product

  $(\vec{u}\times\vec{v})\cdot\vec{w}$. Its magnitude gives the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

It can be evaluated as a determinant: $(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

e.g. (from text) Find the volume of the box determined by $\vec{u} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{v} = -2\hat{i} + 3\hat{k}$ and $\vec{w} = 7\hat{j} - 4\hat{k}$

Solution: $(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = \ldots$

12.5 Lines and planes in space

- In space, a line is determined by a point and a vector giving the direction of the line.

Suppose L is a line through the point $P_0(x_0, y_0, z_0)$ and parallel to $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$. Then L is the collection of all points $P(x, y, z)$ so that $\overrightarrow{P_0P}$ is parallel to $\vec{v}$.

Therefore, $\overrightarrow{P_0P} = t\vec{v}$ for some scalar t.

i.e. $(x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k} = t(v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$

i.e. $x\hat{i} + y\hat{j} + z\hat{k} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k} + t(v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$

If $\vec{r}(t)$ is the position vector of the point $P(x, y, z)$ and $\vec{r}_0(t)$ is the position vector of $P_0(x_0, y_0, z_0)$, then the above equation is:

$$\boxed{\vec{r}(t) = \vec{r}_0(t) + t\vec{v}} \qquad -\infty < t < \infty$$

Vector equation for a line.

Equating the components of the left and right hand side of the equations gives us:

$$\boxed{\begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3 \end{aligned}}$$

Parametric equations for a line

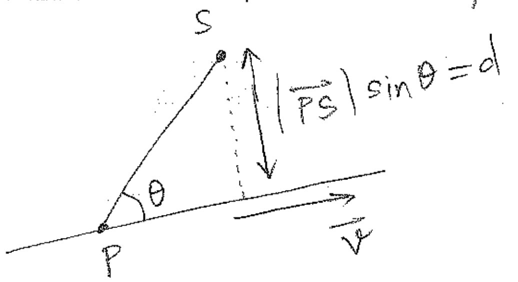
Example Find parametric equations for the line through $(-2,0,4)$ parallel to $\vec{v} = 2\hat{i} + 4\hat{j} - 2\hat{k}$

Solution: Here $(x_0, y_0, z_0) = (-2, 0, 4)$ and $v_1\hat{i} + v_2\hat{j} + v_3\hat{k} = 2\hat{i} + 4j - 2\hat{k}$

so the parametric equations are:

$$x = -2 + 2t$$
$$y = 0 + 4t$$
$$z = 4 - 2t$$

Distance from a point to a line in space



$$d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|}$$

Example Find the distance from the point $S(1, 1, 5)$ to the line L: $x = 1+t$, $y = 3-t$, $z = 2t$

Solution: Step 1: Find a point P on the line, by plugging in some t-value, e.g. $t = 0$ gives $x = 1$, $y = 3$, $z = 0$ in the equation of L. so $P = (1, 3, 0)$ is a point on the line.

Step 2: Find $\overrightarrow{PS} = \langle 0, -2, 5 \rangle$

Step 3: read off $\vec{v} = \langle 1, -1, 2 \rangle$ from the equation for L.

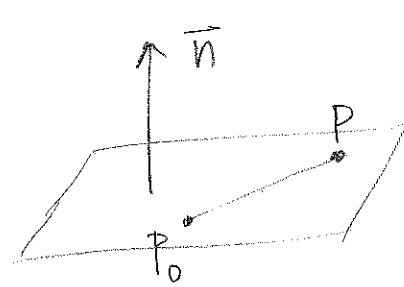
Step 4. Compute $\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix}$

Step 5. Compute $|\overrightarrow{PS} \times \vec{v}|$ and $|\vec{v}|$ and plug-into the above formula!

Equation for a plane in space

A plane is specified by knowing a point on it and a vector normal to the plane.

$P_0(x_0, y_0, z_0)$ and $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$



$P(x, y, z)$ is on the plane iff

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$
$$(A\hat{i} + B\hat{j} + C\hat{k}) \cdot \left((x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k} \right) = 0$$

i.e. $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$

Example. Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\vec{n} = 5\hat{i} + 2\hat{j} - \hat{k}$

Solution: In this case, $(x_0, y_0, z_0) = (-3, 0, 7)$

and $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k} = 5\hat{i} + 2\hat{j} - \hat{k}$

so the equation for the plane is:

$$5(x-(-3)) + 2(y-0) - 1(z-7) = 0$$

i.e. $5(x+3) + 2y - (z-7) = 0$

i.e. $5x + 2y - z = -15 + 7$, i.e. $\boxed{5x + 2y - z = -8}$

Lines of intersection

two planes are parallel iff their normals are parallel. $\vec{n}_1 = k\vec{n}_2$ for some scalar k.

Else, they intersect in a line.

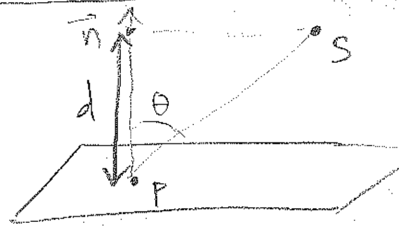
Example. Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution: The line of intersection lies in both the planes, and so it is perpendicular to both normals.

$\vec{n}_1 = \langle 3, -6, -2 \rangle$ and $\vec{n}_2 = \langle 2, 1, -2 \rangle$

a vector perpendicular to $\vec{n}_1$ and $\vec{n}_2$, is $\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \ldots$

Distance from a point to a plane



so $d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$ (Projection of $\vec{PS}$ onto $\vec{n}$)

from the picture,

$d = |\vec{PS}| \cos\theta$

Example. Find the distance of $S(1,1,3)$ from the plane

$3x+2y+6z=6$

Solution: to use the formula on the previous page, we need a point P that lies on the plane.

e.g. $P=(0,0,1)$ satisfies the equation $3x+2y+6z=6$, and hence lies on the plane.

$\overrightarrow{PS}=\langle 1,1,2\rangle$

Also, from the equation of the plane, we can read off

$\vec{n}=\langle 3,2,6\rangle$

so, $\overrightarrow{PS}\cdot\vec{n} = \langle 1,1,2\rangle\cdot\langle 3,2,6\rangle = 3+2+12 = 17$

$|\vec{n}| = \sqrt{3^2+2^2+6^2} = \sqrt{9+4+36} = \sqrt{49} = 7$

so, the distance is $\dfrac{|\overrightarrow{PS}\cdot\vec{n}|}{|\vec{n}|} = \dfrac{17}{7}$.