Lecture 3 Wednesday, 5/25

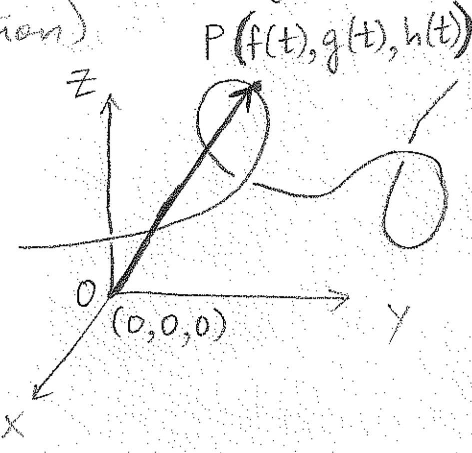
Some topics from 12.5 (see pages 7 and 8 of Lecture 2)

- line of intersection of two planes
- distance of a point to a plane
- angle between planes.

13.1 Vector-valued functions and motion in space

- Suppose there's a particle moving through space during some time interval I (some period of time). Then the x, y and z coordinates of the particle are functions of time t (meaning, they depend on the instant of time under consideration)

$$x = f(t), \quad y = g(t), \quad z = h(t)$$



- The collection of points

$$(x, y, z) = (f(t), g(t), h(t))$$

form a curve in space, which is called the path of the particle.

- The vector that points from the origin to the position $(f(t), g(t), h(t))$ of the particle, is called the position vector $\vec{r}(t)$ of the particle.

$$\vec{r}(t) = \overrightarrow{OP} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

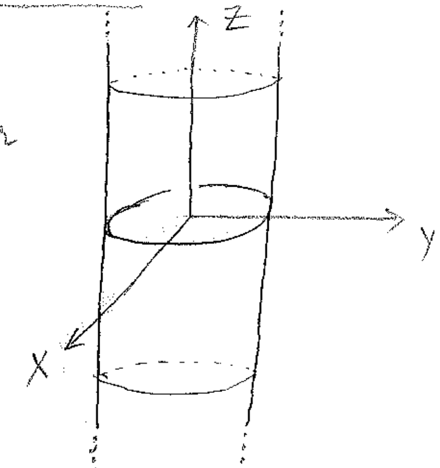
- This is something new! A vector which depends on time. As time varies, the vector $\vec{r}(t)$ points at different points on the curve.

This is called a vector-valued function on the time interval I.

- We can also have a vector-valued function defined on all of 3-space, which gives a vector at each point of 3-space. This is called a <u>vector field</u>.

<u>e.g.</u> the gravitational force field.

- Real-valued functions, for example $f(t)$, $g(t)$ and $h(t)$ from the previous page, are called <u>scalar functions</u>.



- Describing shapes by equations — the cylinder

$$x^2 + y^2 = 1$$

↳ The equation $x^2+y^2=1$ in the x-y plane defines a circle of radius 1, centered at the origin.

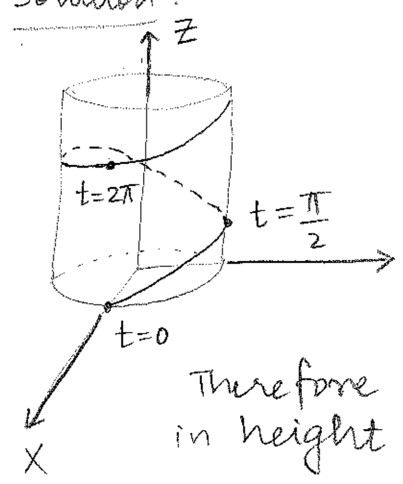
↳ When we consider this equation in 3-space, it means that the x and y coordinates of the point must satisfy the equation, but the z-coordinate can be anything.

↳ The collection of all such points is the cylinder in the picture.

<u>Example</u> (from the textbook)

Draw the graph of the vector function $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$

<u>Solution</u>.



For any t, $\cos^2 t + \sin^2 t = 1$

So the x and y coordinates of points on the curve satisfy the equation $x^2+y^2=1$.

Meanwhile, as t increases, the z-coordinate (which equals t) is increasing.

Therefore the curve goes round the circle and keeps increasing in height. It is a $\boxed{\text{helix}}$.

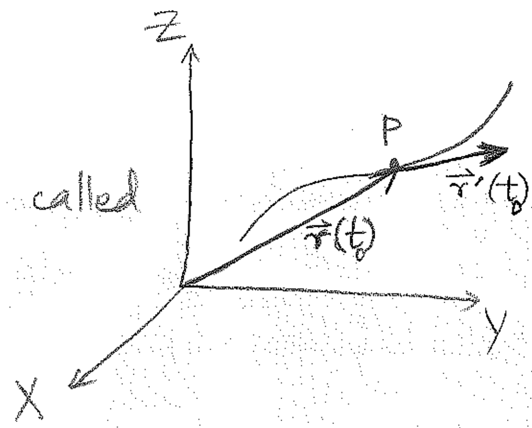
- Derivatives of vector-valued functions

If $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ is the position vector of a particle moving in space, define:

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \frac{df}{dt}\hat{\imath} + \frac{dg}{dt}\hat{\jmath} + \frac{dh}{dt}\hat{k}$$

Example. Suppose $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\hat{k}$ (helix)

then $\vec{r}\,'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}$



- Tangent vector: If $\vec{r}\,'(t_0) \neq \vec{0}$, then $\vec{r}\,'(t_0)$ is called the tangent vector to the curve at P.

- The line passing through P, with direction parallel to $\vec{r}\,'(t_0)$, is called the tangent line to the curve at P

Example. Find the parametric equations for the tangent line to the helix $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}$ at the point $P\left(0, 1, \frac{\pi}{2}\right)$.

Solution: What value of t_0 does this point correspond to?
→ we must have $\cos t_0 = 0$, $\sin t_0 = 1$ and $t_0 = \frac{\pi}{2}$

so $\boxed{t_0 = \frac{\pi}{2}}$

What is $\vec{r}\,'\left(\frac{\pi}{2}\right)$? $\quad \vec{r}\,'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}$

so $\vec{r}\,'\left(\frac{\pi}{2}\right) = -\hat{\imath} + 0\hat{\jmath} + \hat{k} = -\hat{\imath} + \hat{k}$

The tangent line passes through $\left(0, 1, \frac{\pi}{2}\right)$ and has direction $-\hat{\imath} + \hat{k}$
Therefore the parametric equations are:

$$x = 0 - t$$
$$y = 1$$
$$z = \frac{\pi}{2} + t$$

- If $\vec{r}(t)$ = position vector, then $\vec{v}(t) = \frac{d\vec{r}}{dt}$ is called the $\boxed{\text{velocity vector}}$ of the particle. (Speed is the magnitude of velocity, $|\vec{v}(t)|$)

and $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$ is called the $\boxed{\text{acceleration vector}}$

the unit vector $\frac{\vec{v}(t)}{|\vec{v}(t)|}$ is called the direction of motion of the particle at time t.

Example $\vec{r}(t) = (3t+1)\hat{i} + \sqrt{3}t\hat{j} + t^2\hat{k}$ is the position vector of a particle at time t. Find the velocity and acceleration of the particle. Also find the angle between velocity and acceleration vectors at time $t=0$.

Solution. velocity $\vec{v}(t) = \vec{r}\,'(t) = 3\hat{i} + \sqrt{3}\hat{j} + 2t\hat{k}$

acceleration $\vec{a}(t) = \frac{d\vec{v}}{dt} = 0\hat{i} + 0\hat{j} + 2\hat{k} = 2\hat{k}$

velocity at time $t=0$ is $\vec{v}(0) = 3\hat{i} + \sqrt{3}\hat{j} + 0\hat{k} = 3\hat{i} + \sqrt{3}\hat{j}$

acceleration at time $t=0$ is $\vec{a}(0) = 2\hat{k}$

Angle between $\vec{v}(0)$ and $\vec{a}(0)$:

Note that
$$\begin{aligned}\vec{v}(0)\cdot\vec{a}(0) &= (3\hat{i} + \sqrt{3}\hat{j})\cdot 2\hat{k} \\ &= 6(\hat{i}\cdot\hat{k}) + 2\sqrt{3}(\hat{j}\cdot\hat{k}) \\ &= 6\times 0 + 2\sqrt{3}\times 0 \quad (\hat{i}, \hat{j}, \hat{k} \text{ are mutually orthogonal}) \\ &= 0\end{aligned}$$

Therefore $\vec{v}(0)$ and $\vec{a}(0)$ are orthogonal (their dot product is zero) but neither one of them is the zero vector.

Therefore, angle b/w $\vec{v}(0)$ and $\vec{a}(0)$ is 90° $\left(\frac{\pi}{2}\right)$.