MATH 114

Lecture 4 — Thursday, 5/26

13.2 Integrals of Vector functions ; Projectile motion

- If we have a vector function $\vec{r}(t)$ on interval I and $\vec{R}(t)$ is another vector function so that $\frac{d\vec{R}}{dt} = \vec{r}(t)$ then $\vec{R}(t)$ is called an anti-derivative of $\vec{r}(t)$.

- The indefinite integral of $\vec{r}(t)$ is

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

Example: $\int (\sin t\,\hat{i} + e^t\,\hat{j} + \hat{k})\,dt = \left(\int \sin t\,dt\right)\hat{i} + \left(\int e^t\,dt\right)\hat{j} + \left(\int 1\,dt\right)\hat{k}$

$$= (-\cos t + C_1)\hat{i} + (e^t + C_2)\hat{j} + (t + C_3)\hat{k}$$

$$= -\cos t\,\hat{i} + e^t\,\hat{j} + t\hat{k} + (C_1\hat{i} + C_2\hat{j} + C_3\hat{k})$$

$$= -\cos t\,\hat{i} + e^t\,\hat{j} + t\,\hat{k} + \vec{C}$$ [where $\vec{C}$ is the vector $C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$]

- Definite integral:

If $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ where f, g, h are integrable over $[a,b]$ then so is $\vec{r}$ and:

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\hat{i} + \left(\int_a^b g(t)\,dt\right)\hat{j} + \left(\int_a^b h(t)\,dt\right)\hat{k}$$

Example: $\int_{-\pi/4}^{\pi/4} \left[\sin t\,\hat{i} + (1+\cos t)\hat{j} + \sec^2 t\,\hat{k}\right] dt$

$$= \left(\int_{-\pi/4}^{\pi/4} \sin t\,dt\right)\hat{i} + \left(\int_{-\frac{\pi}{4}}^{\pi/4} (1+\cos t)\,dt\right)\hat{j} + \left(\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt\right)\hat{k}$$

$$= \Big[-\cos t\Big]_{-\pi/4}^{\pi/4}\hat{i} + \Big[t + \sin t\Big]_{-\pi/4}^{\pi/4}\hat{j} + \underset{1}{\Big[\tan t\Big]_{-\pi/4}^{\pi/4}}\hat{k} = 0\,\hat{i} + \left(\frac{\pi}{2} + \sqrt{2}\right)\hat{j} + 2\hat{k}$$

$$= \boxed{\left(\frac{\pi}{2} + \sqrt{2}\right)\hat{j} + 2\hat{k}}$$

Example. Suppose we know the acceleration of a particle is:
$\vec{a}(t) = -3\cos t\,\hat{i} - 3\sin t\,\hat{j} + 2\hat{k}$, and we know it starts from the point $(4, 0, 0)$.
We also know the initial velocity of the particle, $\vec{v}(0) = 3\hat{j}$.
Find the position vector of the particle at time t, $\vec{r}(t)$.

Solution. $\vec{a}(t) = \dfrac{d\vec{v}}{dt}$, so $\vec{v}(t) = \int \vec{a}(t)dt + \vec{C}$

i.e. $\vec{v}(t) = \int(-3\cos t\,\hat{i} - 3\sin t\,\hat{j} + 2\hat{k})dt + (C_1\hat{i} + C_2\hat{j} + C_3\hat{k})$

$= -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 2t\,\hat{k} + C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$

therefore $\vec{v}(0) = 0\hat{i} + 3\hat{j} + 0\hat{k} + C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$

$= C_1\hat{i} + (3 + C_2)\hat{j} + C_3\hat{k}$

but we're given that $\vec{v}(0) = 3\hat{j}$

therefore $C_1\hat{i} + (3 + C_2)\hat{j} + C_3\hat{k} = 3\hat{j} = 0\hat{i} + 3\hat{j} + 0\hat{k}$

and so $\boxed{C_1 = 0}$ [By equating coefficients of $\hat{i}, \hat{j}, \hat{k}$]

$3 + C_2 = 3$, so $\boxed{C_2 = 0}$

and $\boxed{C_3 = 0}$

so, $\vec{v}(t) = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 2t\,\hat{k}$

now, $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$, so $\vec{r}(t) = \int \vec{v}(t)dt + \vec{D}$

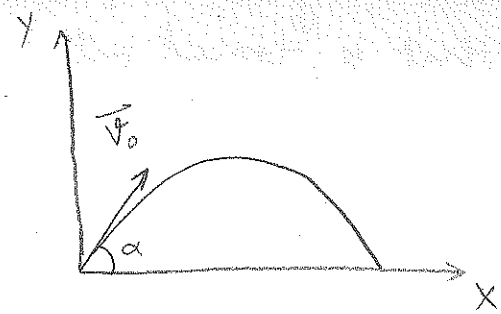
i.e. $\vec{r}(t) = \int(-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 2t\,\hat{k})dt + D_1\hat{i} + D_2\hat{j} + D_3\hat{k}$

$= 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + t^2\hat{k} + D_1\hat{i} + D_2\hat{j} + D_3\hat{k}$

$= (3\cos t + D_1)\hat{i} + (3\sin t + D_2)\hat{j} + (t^2 + D_3)\hat{k}$

therefore $\vec{r}(0) = (3 + D_1)\hat{i} + D_2\hat{j} + D_3\hat{k}$
and so the starting point of the particle is $(3 + D_1, D_2, D_3)$
But we know this is $(4, 0, 0)$.
therefore $D_1 = 1, D_2 = 0, D_3 = 0$, and so $\boxed{\vec{r}(t) = (3\cos t + 1)\hat{i} + 3\sin t\,\hat{j} + t^2\hat{k}}$

• Projectile motion



An object is fired from an initial position on the ground, at some angle α with the horizontal (X axis)

The only force acting on it is gravity.

We want to describe its motion.

call the initial velocity $\vec{v}_0$ and initial position $\vec{r}_0$.

then $\vec{r}_0 = 0\hat{i} + 0\hat{j} = \vec{0}$

and $\vec{v}_0 = |\vec{v}_0|\cos\alpha\,\hat{i} + |\vec{v}_0|\sin\alpha\,\hat{j}$

Note. α is called the launch angle or firing angle or angle of elevation

to simplify notation, denote $|\vec{v}_0|$ by u_0 (u_0 is the initial speed)

i.e. $\vec{v}_0 = u_0\cos\alpha\,\hat{i} + u_0\sin\alpha\,\hat{j}$

Now: mass x acceleration = force

so $m \cdot \dfrac{d^2\vec{r}}{dt^2} = -mg\hat{j}$

so $\dfrac{d^2\vec{r}}{dt^2} = -g\hat{j}$

integrating, we get $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = -gt\,\hat{j} + \vec{v}_0$

and integrating again, $\vec{r}(t) = \dfrac{-gt^2}{2}\hat{j} + \vec{v}_0 t + \vec{r}_0$.

so $\vec{r}(t) = \dfrac{-1}{2}gt^2\hat{j} + tu_0\cos\alpha\,\hat{i} + tu_0\sin\alpha\,\hat{j} + \vec{0}$

$= tu_0\cos\alpha\,\hat{i} + \left(tu_0\sin\alpha - \frac{1}{2}gt^2\right)\hat{j}$

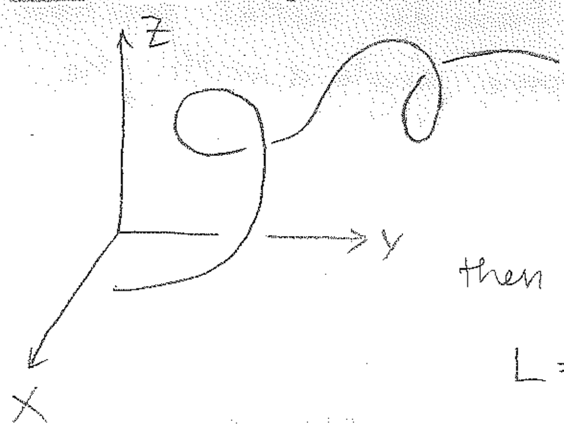
Some facts we can derive from the above equation:

1. Maximum height, $\boxed{y_{max} = \dfrac{(u_0\sin\alpha)^2}{2g}}$

2. time of flight, $\boxed{T = \dfrac{2v_0\sin\alpha}{g}}$

3. maximum horizontal distance (range), $\boxed{R = \dfrac{u_0^2}{g}\sin 2\alpha}$

13.3 Arc length in space



curve $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ for $a \le t \le b$ that is traced out exactly once as t goes from $t=a$ to $t=b$

then length of the curve is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$$

Alternately, $L = \int_a^b |\vec{v}(t)|\, dt$

Example: What is the length of the helix $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$ from $t=0$ to $t = 2\pi$?

Solution.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$$

$$|\vec{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{1+1} = \sqrt{2}$$

so $L = \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\pi\sqrt{2}}$

Arc length parameter (with base point $P(t_0)$)

$$s(t) = \int_{t_0}^{t} \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2}\; d\tau = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau$$

this formula gives the length of the curve from the base point up until the point $\vec{r}(t)$.

Example. helix $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$, $t_0 = 0$

as computed above, $|\vec{v}(\tau)| = \sqrt{2}$

so $s(t) = \int_0^t \sqrt{2}\, d\tau = \sqrt{2}\, t$, so $t = \frac{s}{\sqrt{2}}$

Then, $\vec{r}(t(s)) = \left(\cos\left(\frac{s}{\sqrt{2}}\right)\right)\hat{i} + \left(\sin\left(\frac{s}{\sqrt{2}}\right)\right)\hat{j} + \frac{s}{\sqrt{2}}\hat{k}$ is called an "arc-length parametrization" for the helix.

• Speed: $\frac{ds}{dt} = |\vec{v}(t)|$

• Unit tangent vector $\vec{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$ is a unit vector tangent to the curve. It is called the unit tangent vector to the curve at $\vec{r}(t)$.

Example. $\vec{r}(t) = t^3\hat{i} - 2t^3\hat{j} - 3t^3\hat{k}$. Find the unit tangent vector.

Solution: $\vec{v}(t) = 3t^2\hat{i} - 6t^2\hat{j} - 9t^2\hat{k}$

so
$$|\vec{v}(t)| = \sqrt{(3t^2)^2 + (-6t^2)^2 + (-9t^2)^2}$$
$$= \sqrt{9t^4 + 36t^4 + 81t^4}$$
$$= \sqrt{126t^4}$$
$$= \sqrt{126}\, t^2$$
$$= 3\sqrt{14}\, t^2$$

so
$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{3t^2\hat{i} - 6t^2\hat{j} - 9t^2\hat{k}}{3\sqrt{14}\,t^2}$$
$$= \frac{1}{\sqrt{14}}\hat{i} - \frac{2}{\sqrt{14}}\hat{j} - \frac{3}{\sqrt{14}}\hat{k}$$