MATH 114

Lecture 5 Tuesday, 5/31

Some more topics from Section 13.1

recall: in a previous lecture we defined vector functions and saw how to differentiate them. Differentiation of vector functions follows some important rules:

Let $\vec{u}(t)$ and $\vec{v}(t)$ be differentiable vector functions, $\vec{C}$ a constant vector, c a scalar (constant number) and f a differentiable function.

1. $\frac{d\vec{C}}{dt} = 0$

2. $\frac{d}{dt}\left(c\vec{u}(t)\right) = c\vec{u}'(t)$

3. $\frac{d}{dt}\left(\vec{u}(t) \pm \vec{v}(t)\right) = \vec{u}'(t) \pm \vec{v}'(t)$

4. Dot product rule: $\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

5. Cross product rule: $\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

6. Chain rule: $\frac{d}{dt}\left(\vec{u}(f(t))\right) = f'(t)\,\vec{u}'(f(t))$

Example: $\vec{u}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t^2\hat{k}$. Let $\vec{v}(t) = \vec{u}(t^2+2)$.

What is $\vec{v}'(t)$?

Solution: (using chain rule). $\vec{u}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + 2t\,\hat{k}$

$\vec{v}'(t) = \frac{d}{dt}\vec{v}(t) = \frac{d}{dt}\left[\vec{u}(t^2+2)\right] = ?$

let $f(t) = t^2+2$, then $\frac{d}{dt}\left[\vec{u}(t^2+2)\right] = f'(t).\,\vec{u}'(f(t))$

$f'(t) = 2t$

$$= 2t.\,\vec{u}'(t^2+2)$$

$$= 2t\cdot\left(-\sin(t^2+2)\,\hat{i} + \cos(t^2+2)\,\hat{j} + 2(t^2+2)\,\hat{k}\right)$$

• An application of the <u>dot product rule</u> : <u>vector functions of constant length</u>.

Suppose we're given that $\vec{r}(t)$ has constant magnitude a.

i.e. $|\vec{r}(t)|^2 = a^2$

so $\vec{r}(t) \cdot \vec{r}(t) = a^2$

Differentiate both sides: $\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$

(since a^2 is a constant).

So, $2\,\vec{r}\,'(t) \cdot \vec{r}(t) = 0$

so $\boxed{\vec{r}\,'(t) \cdot \vec{r}(t) = 0}$

That is, if $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ and its derivative $\vec{r}\,'(t)$ are orthogonal.

• An application of the <u>chain rule</u> :

recall that at the end of the last lecture we talked about the arc length parameter, and the arc length parametrization of the helix, which was: $\vec{r}(t(s)) = \cos\left(\frac{s}{\sqrt{2}}\right)\hat{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\hat{j} + \frac{s}{\sqrt{2}}\hat{k}$

we also noted that $\frac{ds}{dt} = |\vec{v}(t)|$ (Speed)

Then, so long as we consider curves with speed not equal to zero, we have $\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\vec{v}|}$

Then, thinking of $\vec{r}$ as a function of s,

$$\frac{d\vec{r}}{ds} = \frac{d}{ds}\left(\vec{r}(t(s))\right) = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \vec{v}(t) \cdot \frac{1}{|\vec{v}|} = \frac{\vec{v}}{|\vec{v}|} = \vec{T}$$

That is, $\frac{d\vec{r}}{ds}$ is the unit tangent vector of the curve.

13.4 Curvature and Normal vectors of a curve

- As a particle moves along a curve, its direction of motion, given by the unit tangent vector $\vec{T}$, is changing. The rate at which $\vec{T}$ changes per unit length of the curve, is called the <u>curvature</u>, κ.

$$\boxed{\kappa = \left|\frac{d\vec{T}}{ds}\right|}$$

- The quantity κ tells you how sharply the curve is turning at a given point on the curve.

- Note: by the chain rule, $\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}\right|$

$$= \left|\frac{d\vec{T}}{dt}\right| \cdot \left|\frac{dt}{ds}\right| = \left|\frac{d\vec{T}}{dt}\right| \cdot \frac{1}{|\vec{v}|}$$

so $$\boxed{\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|}$$

<u>Example</u>. $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ is the vector equation for a line.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\vec{r}_0 + t\vec{v}) = \frac{d}{dt}\vec{r}_0 + \frac{d}{dt}(t\vec{v}) = 0 + \vec{v} = \vec{v}$$

a constant vector

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}}{|\vec{v}|}$$ ← a constant unit vector in the direction of $\vec{v}$

Therefore $\frac{d\vec{T}}{dt} = \vec{0}$

so, curvature $\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{|\vec{v}|}\cdot|\vec{0}| = \boxed{0}$

That is, the curvature of a straight line is zero.

Example: $\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$ parametrizes a circle of radius a in the x-y plane.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$$

$$|\vec{v}(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$$

so $$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{-a\sin t\,\hat{i} + a\cos t\,\hat{j}}{a} = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

and $$\frac{d\vec{T}}{dt} = -\cos t\,\hat{i} - \sin t\,\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1$$

therefore, curvature $$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{a}\cdot 1 = \boxed{\frac{1}{a}}$$

A circle has constant curvature equalling $\frac{1}{\text{radius}}$.

- Unit normal vector for a curve

$$\boxed{\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}}$$

Since $\kappa = \left|\frac{d\vec{T}}{ds}\right|$, $\vec{N}$ is indeed a unit vector.

Also, since $\vec{T}$ is a vector function of constant length $(=1)$, $\vec{T}$ is orthogonal to its derivative $\frac{d\vec{T}}{ds}$.

So $\vec{N}$ is a unit vector defined along the curve, which at each point on the curve is orthogonal to the unit tangent vector at that point.

• Formula for calculating $\vec{N}$:

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{d\vec{T}/ds}{|d\vec{T}/ds|} = \frac{\frac{d\vec{T}}{dt}\cdot \frac{dt}{ds}}{\left|\frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}\right|}$$

$$= \frac{d\vec{T}/dt \cdot dt/ds}{|d\vec{T}/dt|\;|dt/ds|} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

so $$\boxed{\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}}$$

Example: helix $\vec{r}(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 4t\,\hat{k}$

Find the curvature κ and the unit normal vector $\vec{N}$

Solution: $\vec{v}(t) = \frac{d\vec{r}}{dt} = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}$

$$|\vec{v}(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = \sqrt{3^2+4^2} = 5$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = -\frac{3}{5}\sin t\,\hat{i} + \frac{3}{5}\cos t\,\hat{j} + \frac{4}{5}\hat{k}$$

$$\frac{d\vec{T}}{dt} = -\frac{3}{5}\cos t\,\hat{i} - \frac{3}{5}\sin t\,\hat{j}$$

curvature $$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \frac{1}{5}\cdot\left|-\frac{3}{5}\cos t\,\hat{i} - \frac{3}{5}\sin t\,\hat{j}\right| = \frac{1}{5}\cdot\sqrt{\left(\frac{-3}{5}\cos t\right)^2 + \left(-\frac{3}{5}\sin t\right)^2}$$

$$= \frac{1}{5}\cdot\frac{3}{5} = \boxed{\frac{3}{25}}$$

unit normal vector $$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{-\frac{3}{5}\cos t\,\hat{i} - \frac{3}{5}\sin t\,\hat{j}}{\frac{3}{5}}$$

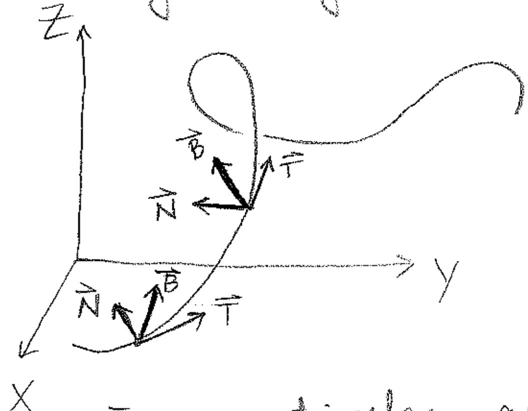
$$= \boxed{-\cos t\,\hat{i} - \sin t\,\hat{j}}$$

13.5 Tangential and Normal components of acceleration

- Binormal vector $\vec{B} = \vec{T} \times \vec{N}$ (Definition)

is a unit vector defined at each point of the curve, that is orthogonal to both $\vec{T}$ (unit tangent vector) and $\vec{N}$ (unit normal vector)

- together $\vec{T}$, $\vec{N}$ and $\vec{B}$ give rise to a vector frame that is moving along the curve. This is called the TNB frame.



- Any vector that is defined at each point of the curve, can be written in terms of its $\vec{T}$, $\vec{N}$ and $\vec{B}$ components, just like you would usually write a vector in terms of its $\hat{i}$, $\hat{j}$, $\hat{k}$ components.

In particular, acceleration.

Note that $\vec{v} = |\vec{v}| \cdot \vec{T} = \frac{ds}{dt} \cdot \vec{T}$

Differentiating, $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\vec{T}\right) = \frac{d^2s}{dt^2} \cdot \vec{T} + \frac{ds}{dt} \cdot \frac{d\vec{T}}{dt}$

$$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt} \cdot \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$$

$$= \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \cdot \frac{d\vec{T}}{ds} = \underline{\frac{d^2s}{dt^2}}\,\vec{T} + \underline{\left(\frac{ds}{dt}\right)^2 \cdot \kappa}\,\vec{N}$$

So, the acceleration vector has a $\vec{T}$-component and an $\vec{N}$-component but no $\vec{B}$-component.

The $\vec{T}$-component is $\boxed{a_T = \frac{d^2s}{dt^2}}$ (tangential component of acceleration)

The $\vec{N}$-component is $\boxed{a_N = \left(\frac{ds}{dt}\right)^2 \cdot \kappa}$ (normal component of acceleration)