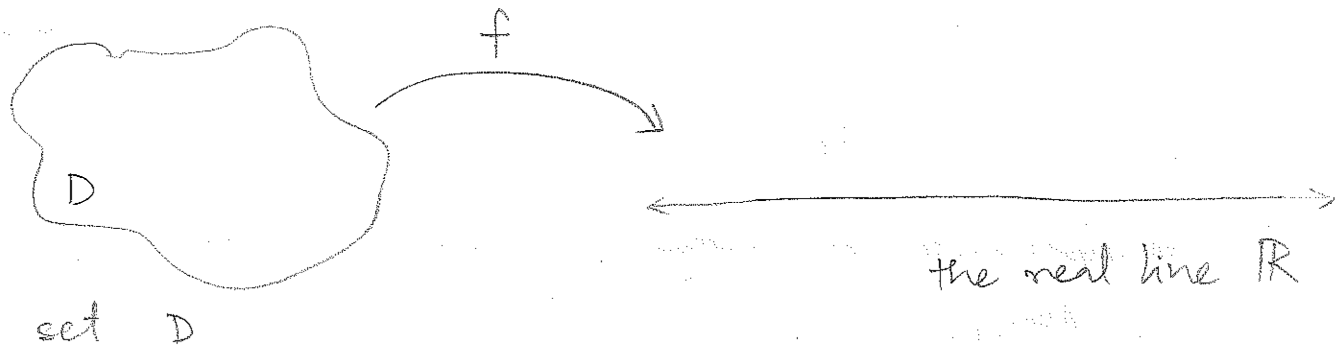
Lecture 6 Wednesday 6/1

14.1 Functions of several variables

- A real-valued function f is a rule, which for each element in a given set D, assigns a unique real number.



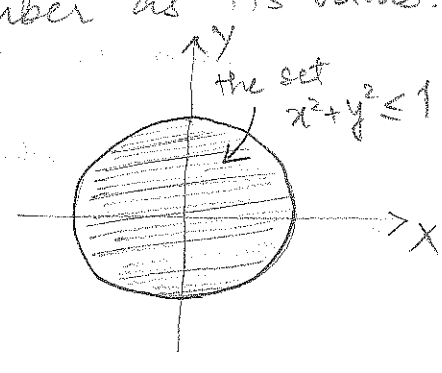
- the set D is called the domain of f, and the collection of real numbers that appear as values of the function, is called the range of f.
- If the domain D consists of pairs of numbers (x, y) then f is called a function of two variables.
- If the domain D consists of 3-tuples (x, y, z) then f is called a function of three variables.
- Finding the domain and range of a function

If the function f is given by a formula, then the domain consists of all elements (x,y) or (x,y,z) for which the formula "makes sense", i.e. gives a real number as its value.

E.g. $f(x,y) = \sqrt{1-x^2-y^2}$

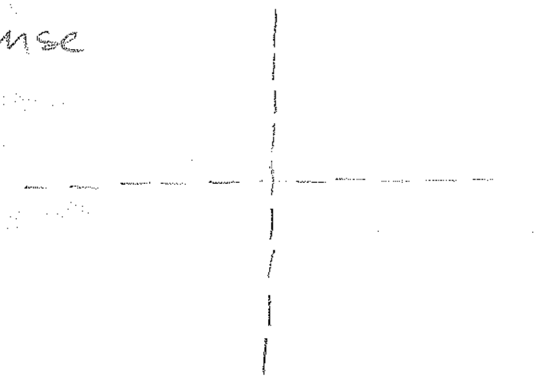
This makes sense only when $1-x^2-y^2 \geq 0$

i.e. when $x^2+y^2 \leq 1$



E.g. $f(x,y) = \dfrac{1}{xy}$ This makes sense when $xy \neq 0$

i.e. when neither x nor y is zero.



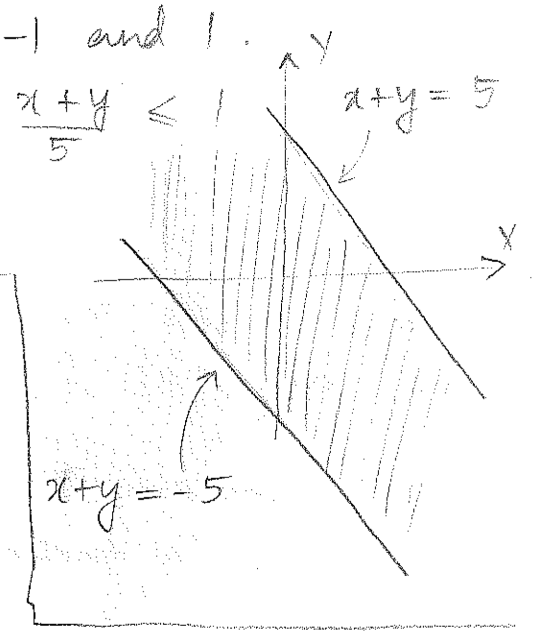
the X-Y plane with the X and Y axes deleted.

E.g. $g(x,y) = \sin^{-1}\left(\dfrac{x+y}{5}\right)$

$\sin^{-1}$ is defined only for values between -1 and 1.

therefore the domain is given by $-1 \leq \dfrac{x+y}{5} \leq 1$

i.e. $-5 \leq x+y \leq 5$



E.g. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

domain = all of 3-space

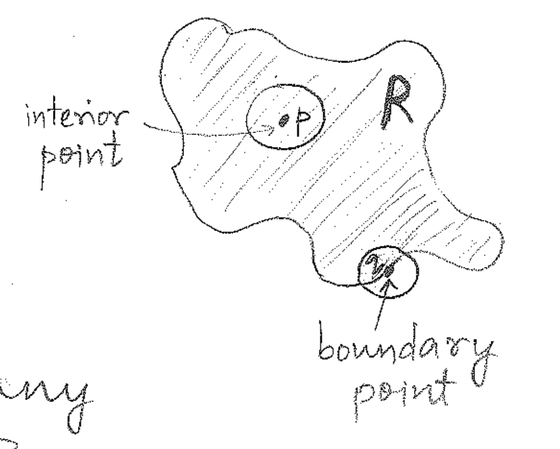
range = $[0, \infty)$

E.g. $f(x,y,z) = \ln z + \sqrt{y-x^2}$

domain = (the half-space $z>0$) intersected with the region ($y \geq x^2$)

Some definitions

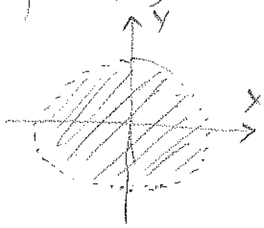
For a region R in the xy plane, a point $p = (x_0, y_0)$ is called an interior point if it's possible to make a small disk centered at p, which lies completely within R.



$q = (x_1, y_1)$ is called a boundary point if any disk centered at q contains points of R as well as points outside R.

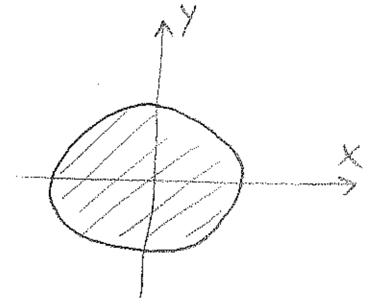
- If R consists of only interior points (no boundary points) we say R is an <u>open set</u>.

<u>e.g.</u> the set $x^2+y^2<1$ (the interior of the disk)



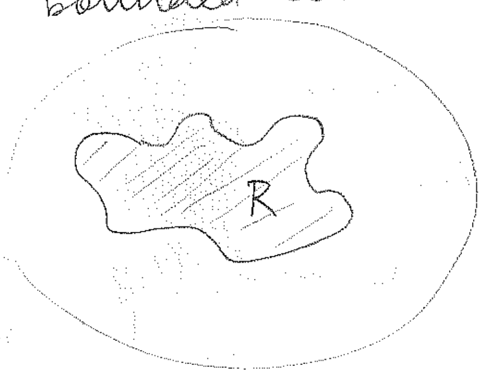
- If R contains all its boundary points we say R is a <u>closed set</u>

<u>e.g.</u> the set $x^2+y^2 \leq 1$ (the disk along with its boundary)

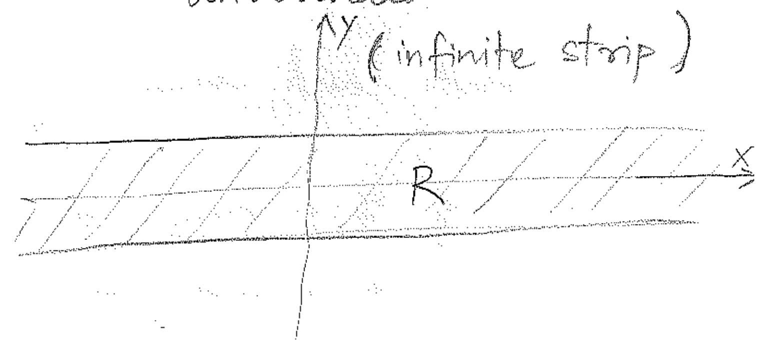


- If R is contained inside a disk of finite radius we say R is <u>bounded</u>. If not, we say R is <u>unbounded</u>.

<u>e.g.</u> bounded set:



unbounded set (infinite strip)

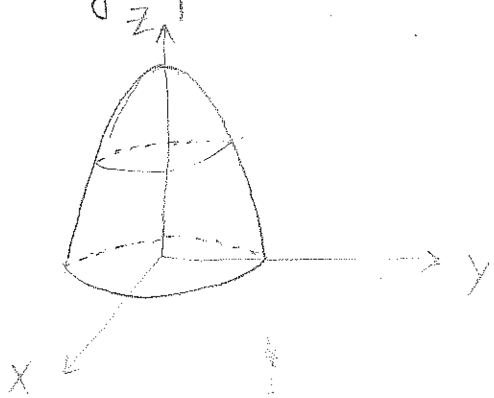


- A function of two variables can also be represented as a <u>graph</u>. This is just the collection of points $(x, y, f(x,y))$

It is also called the surface $z = f(x,y)$.

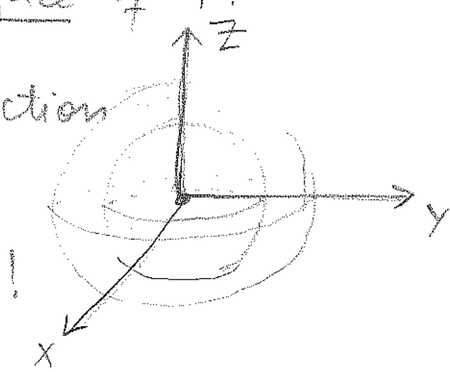
<u>e.g.</u> $f(x,y) = 10 - x^2 - y^2$

Its graph is what's called a <u>paraboloid</u>



- On the other hand, let's think about functions of three variables $f(x, y, z)$.
- The set of points (x,y,z) in 3-space where f has a constant value $f(x, y, z) = c$, is called a <u>level surface</u> of f.

<u>E.g.</u> What are the <u>level surfaces</u> of the function
$$f(x,y,z) = \sqrt{x^2+y^2+z^2} \ ?$$



$\rightarrow$ They are spheres centered at the origin!

- <u>Note</u>. We can define the notions of <u>interior</u>, <u>boundary</u>, <u>open</u> and <u>closed</u> for regions in 3-space analogous to the definitions for regions in the xy plane.

<u>14.2</u> <u>Limits and Continuity in Higher dimensions</u>

- <u>The meaning of a limit of a function $f(x,y)$</u>

If there is a number L so that as the point (x,y) gets very close to (x_0, y_0), the value $f(x,y)$ approaches L, then we say
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$
"the limit of $f(x,y)$ as (x,y) approaches / tends to (x_0, y_0), equals L"

- It is important that (x,y) is allowed to approach (x_0, y_0) in all directions!

· Properties

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$, then

1. $\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y) \pm g(x,y) \right) = L \pm M$

2. $\lim_{(x,y)\to(x_0,y_0)} k f(x,y) = kL$

3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) \cdot g(x,y) = L \cdot M$

4. $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ provided $M \neq 0$

5. $\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y) \right)^n = L^n$ if n a positive integer

6. $\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y) \right)^{\frac{1}{n}} = L^{\frac{1}{n}}$, n a positive integer, and if n even, we require $L > 0$.

Example.

① $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = \boxed{5}$

② $\lim_{(x,y)\to(1,2)} \dfrac{x^3+3y-6}{2x-y^3} = \dfrac{1^3+3\times 2-6}{2\times 1-2^3} = \dfrac{1+6-6}{2-8} = \dfrac{1}{-6}$

$= \boxed{-\dfrac{1}{6}}$

③ $\lim_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$ ... Plugging-in doesn't work because we get $\dfrac{0}{0}$.

so, try to factorize: $\dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \dfrac{x(x-y)}{\sqrt{x} - \sqrt{y}} = \dfrac{x\left((\sqrt{x})^2 - (\sqrt{y})^2\right)}{\sqrt{x} - \sqrt{y}}$

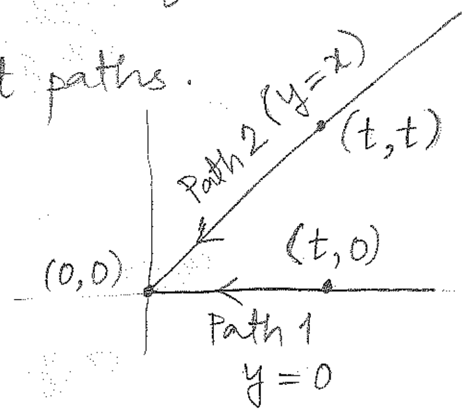
$= \dfrac{x\,\cancel{(\sqrt{x} - \sqrt{y})}\,(\sqrt{x} + \sqrt{y})}{\cancel{\sqrt{x} - \sqrt{y}}} = x(\sqrt{x} + \sqrt{y})$

so the limit becomes $\displaystyle\lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) = 0.$

④ Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^3 + y^3}$ exist?

Again, plugging-in doesn't help since it gives $\frac{0}{0}$.

Let's try approaching $(0,0)$ along different paths.



Path 1: approach along the x-axis, i.e. through points $(t,0)$.

then, we get $\dfrac{2 \times t^2 \times 0}{t^3 + 0^3} = \boxed{0}$

Path 2: approach along the path $y = x$, i.e. through points (t,t).

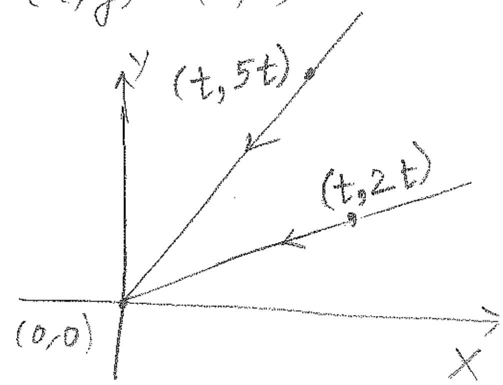
then, we get $\dfrac{2 \times t^2 \times t}{t^3 + t^3} = \dfrac{2t^3}{2t^3} = \boxed{1}$

Since the function approaches different values along different paths, the limit does not exist!

- A function $f(x,y)$ is continuous at (x_0, y_0) if
  1. f is defined at (x_0, y_0)
  2. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
  3. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

A function is continuous if it is continuous at every point in its domain

Example. Show that
$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$
is not continuous at $(0,0)$.



Solution. We'll show that the limit
$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist}$$

Path 1. approach through points $(t, 2t)$ (line of slope 2)

then $$\frac{2xy}{x^2+y^2} = \frac{2 \times t \times 2t}{t^2+(2t)^2} = \frac{4t^2}{t^2+4t^2} = \frac{4t^2}{5t^2} = \boxed{\frac{4}{5}}$$

Path 2. approach through points $(x, 5x)$ (line of slope 5)

then $$\frac{2xy}{x^2+y^2} = \frac{2 \times t \times 5t}{t^2+(5t)^2} = \frac{10t^2}{t^2+25t^2} = \frac{10t^2}{26t^2} = \boxed{\frac{5}{13}}$$

Since the function approaches different values along different paths, the limit does not exist!

So f is not continuous at $(0,0)$.