MATH 114

Lecture 7   Thursday 6/2

14.3 Partial derivatives

- A derivative is a "rate of change". More concretely, for a function $f(x)$ of one variable, it's derivative is defined as a limit

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

That is, it's the limit of the ratio of a small change in the function and a small change Δx in the variable x, as Δx tends to zero.

- For a function of more than one variable, $f(x,y)$ (or $g(x,y,z)$) changing either x or y (or z) results in a change in the value of the function.
- So we can talk about the rate of change of f with respect to the variable x. That is, keep y at some fixed value, and vary x, and see what happens to $f(x,y)$.
- We define the partial derivative of f with respect to x, as:

$$f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

- similarly, we define the partial derivative of f with respect to y:

$$f_y = \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

- And similarly for partial derivative with respect to z, if we're considering a function of three variables $g(x,y,z)$.

- In practice, the way to compute a partial derivative with respect to (w.r.t.) x, is: pretend the other variables y and z are constants, and do the derivative just as you would ordinarily do if it were a function <u>only of the variable x</u>.

<u>Example:</u>

① $f(x,y) = x^2y^3 + 2x + e^y$

$$\frac{\partial f}{\partial x} = 2xy^3 + 2 + 0 = 2xy^3 + 2$$

$$\frac{\partial f}{\partial y} = x^2 \cdot 3y^2 + 0 + e^y = 3x^2y^2 + e^y$$

② $f(x,y) = \sin(xy)$

$$\frac{\partial f}{\partial x} = \cos(xy) \cdot \frac{\partial}{\partial x}(xy) = \cos(xy) \cdot y = y\cos(xy)$$

this step is the chain rule

$$\frac{\partial f}{\partial y} = \cos(xy) \cdot \frac{\partial}{\partial y}(xy) = \cos(xy) \cdot x = x\cos(xy)$$

③ $g(x,y,z) = y^2e^x + \ln z + zy$

$$\frac{\partial g}{\partial x} = y^2e^x + 0 + 0 = y^2e^x$$

$$\frac{\partial g}{\partial y} = 2y \cdot e^x + 0 + z = 2ye^x + z$$

$$\frac{\partial g}{\partial z} = 0 + \frac{1}{z} + y = \frac{1}{z} + y$$

- <u>Second order partial derivatives</u>

What we get when we differentiate a function twice.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad ; \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad ; \quad f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

<u>Example</u> $f(x, y) = e^{xy} + y\cos x$

$$\frac{\partial f}{\partial x} = ye^{xy} - y\sin x \quad ; \quad \frac{\partial f}{\partial y} = xe^{xy} + \cos x$$

$$\frac{\partial^2 f}{\partial x} = \frac{\partial}{\partial x}\left(ye^{xy} - y\sin x\right) = y^2 e^{xy} - y\cos x$$

$$\frac{\partial^2 f}{\partial y} = \frac{\partial}{\partial y}\left(xe^{xy} + \cos x\right) = x^2 e^{xy} + 0 = x^2 e^{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(xe^{xy} + \cos x\right) = xye^{xy} + 1\cdot e^{xy} - \sin x = xye^{xy} + e^{xy} - \sin x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(ye^{xy} - y\sin x\right) = y\cdot xe^{xy} + 1\cdot e^{xy} - \sin x = xye^{xy} + e^{xy} - \sin x$$

<u>Theorem</u>: If f and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open set R and are all continuous, then

$$f_{xy} = f_{yx}$$

[Mixed partial derivatives are equal]

Notice that this agrees with what we computed in the previous example.

14.4 The chain rule

Recall from single variable calculus: the chain rule is what we use when differentiating something that is a "function of a function".

Say f is a function of x, $f(x)$ and x is a function of t, $x(t)$. This makes f a function of t, $f(x(t))$, and:

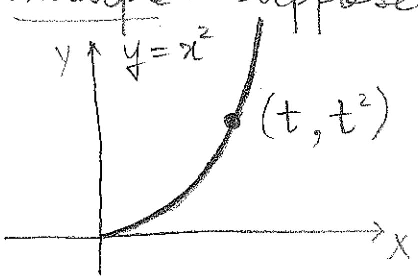
$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}$$

- Now suppose we're looking at a function of two variables, $w = f(x,y)$, where both x and y are functions of t; $x(t)$ and $y(t)$. This makes w a function of t.

Then, the chain rule states that:

$$\frac{dw}{dt} = \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Example. suppose $w = f(x,y) = x^2 y$, where $x(t) = t$, $y(t) = t^2$



That is, (x,y) is moving along the parabola $y = x^2$. How does w change as we move along this path?

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial}{\partial x}(x^2 y) \cdot \frac{d}{dt}(t) + \frac{\partial}{\partial y}(x^2 y) \cdot \frac{d}{dt}(t^2)$$

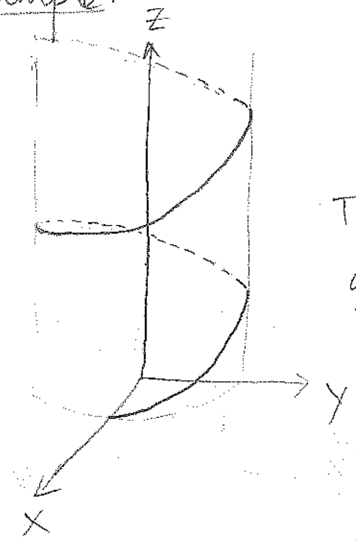
$$= (2xy) \cdot 1 + (x^2) \cdot 2t$$

$$= 2 \times t \times t^2 + t^2 \times 2t = 2t^3 + 2t^3 = \boxed{4t^3}$$

- For a function of three variables, $\omega = f(x, y, z)$ where $x = x(t)$, $y = y(t)$, $z = z(t)$ are all functions of t,

$$\frac{dw}{dt} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Example.



$\omega = xyz$, and the point (x, y, z) is moving along the helix:

$$x(t) = \cos t,\ y(t) = \sin t,\ z(t) = t$$

Then,

$$\frac{dw}{dt} = \frac{\partial(xyz)}{\partial x}\frac{d\cos t}{dt} + \frac{\partial(xyz)}{\partial y}\frac{d\sin t}{dt} + \frac{\partial(xyz)}{\partial z}\frac{dt}{dt}$$

$$= (yz)(-\sin t) + (xz)\cos t + xy \cdot 1$$

$$= (\sin t \times t)(-\sin t) + (\cos t \cdot t)\cos t + (\cos t \times \sin t)$$

$$= \boxed{-t\sin^2 t + t\cos^2 t + \sin t\cos t}$$

- Now, what if we have $\omega = f(x, y, z)$, with each of x, y and z themselves being functions of two variables r and s. i.e. $x = x(r, s)$, $y = y(r, s)$, $z = z(r, s)$. This makes ω a function of r and s:

$$\omega(r, s) = f(x(r,s), y(r,s), z(r,s))$$

So we can ask, what are the partial derivatives of ω with respect to r and s? Once again, these are given by the chain rule:

$$\frac{\partial w}{\partial r} = \frac{\partial \omega}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example. $w = x + yz$, $x = r^2 s$, $y = r + s$, $z = \sin s$

Then, $\dfrac{\partial w}{\partial r} = \dfrac{\partial (x+yz)}{\partial x}\dfrac{\partial (r^2 s)}{\partial r} + \dfrac{\partial}{\partial y}(x+yz)\dfrac{\partial (r+s)}{\partial r} + \dfrac{\partial}{\partial z}(x+yz)\dfrac{\partial}{\partial r}(\sin s)$

$$= 1 \times 2rs + z \times 1 + y \times 0$$

$$= \boxed{2rs + \sin s}$$

$$\frac{\partial w}{\partial s} = \frac{\partial}{\partial x}(x+yz)\frac{\partial (r^2 s)}{\partial s} + \frac{\partial}{\partial y}(x+yz)\frac{\partial (r+s)}{\partial s} + \frac{\partial}{\partial z}(x+yz)\frac{\partial}{\partial s}(\sin s)$$

$$= 1 \times r^2 + z \times 1 + y \times \cos s$$

$$= \boxed{r^2 + \sin s + (r+s)\cos s}$$