MATH 114

Lecture 8 Tuesday 6/7

Directional derivatives

- Last class we learnt about the partial derivatives of a function $f(x,y)$ with respect to x and y, i.e. $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
- $\frac{\partial f}{\partial x}$ tells us about the change in the function value if we vary x while keeping y fixed. That is, it is talking about the rate of change of the function as we move in the direction of the x-axis.
- Similarly $\frac{\partial f}{\partial y}$ is talking about the rate of change of the function in the y-direction.
- More generally, we can speak about the rate of change of the function $f(x,y)$ in the direction of the unit vector $\vec{u} = u_1\hat{i} + u_2\hat{j}$
- The directional derivative of $f(x,y)$ along $\vec{u}$, or the derivative of f in the direction $\vec{u}$ at the point $P_0(x_0, y_0)$ is defined as:

$$\left(\frac{df}{ds}\right)_{\vec{u},\,P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1\,,\, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists.

- Now we're going to see a formula to compute directional derivatives. But first we need this definition:

Gradient: for a real valued function $f(x,y)$, its gradient vector ∇f (read as "grad f") is a vector defined as:

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

Similarly, for a function of three variables $g(x,y,z)$, its gradient is defined as:

$$\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$$

Example ① $f(x,y) = x^2 y$

Then, $\frac{\partial f}{\partial x} = 2xy$; $\frac{\partial f}{\partial y} = x^2$

$$\Rightarrow \boxed{\nabla f = 2xy\,\hat{i} + x^2\hat{j}}$$

② $g(x,y,z) = xyz$

Then $\frac{\partial g}{\partial x} = yz$; $\frac{\partial g}{\partial y} = xz$; $\frac{\partial g}{\partial z} = xy$

$$\Rightarrow \boxed{\nabla g = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}}$$

- Now lets see a formula to compute the directional derivative of a function, using its gradient.

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$

Example. Find the derivative of the function

$f(x,y) = x^2 y + \sin(xy)$ at the point $(1,0)$, in the direction of $\vec{v} = \hat{i} - \hat{j}$

Solution. First we find the unit vector $\vec{u}$ in the direction of $\vec{v}$, by dividing $\vec{v}$ by its magnitude $|\vec{v}|$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{i} - \hat{j}}{\sqrt{1^2+(-1)^2}} = \frac{\hat{i} - \hat{j}}{\sqrt{2}} = \frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}$$

Next, we need ∇f at the point $(1,0)$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = (2xy + y\cos(xy))\hat{i} + (x^2 + x\cos(xy))\hat{j}$$

$$(\nabla f)_{(1,0)} = (2\times 1\times 0 + 0\times\cos(1\times 0))\hat{i} + (1^2 + 1\times\cos(1\times 0))\hat{j}$$

$$= 0\hat{i} + 2\hat{j} = 2\hat{j}$$

$$\Rightarrow \left(\frac{df}{ds}\right)_{\vec{u},(1,0)} = 2\hat{j}\cdot\left(\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}\right) = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

- Properties of the gradient

The gradient of f at a point P_0, $(\nabla f)_{P_0}$ is the direction in which the directional derivative of f at that point, is maximum. i.e., it is the direction of fastest increase of f at P_0.

On the other hand, the vector $-(\nabla f)_{P_0}$ gives the direction of fastest decrease of f at P_0.

- $\boxed{\text{EXAMPLE}}$ ....
- Recall that we talked about the level surface of a function $g(x,y,z)$ of three variables. And similarly, we have a level curve for a function of two variables $f(x,y)$.

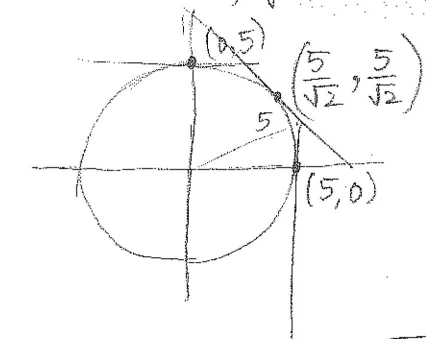
i.e., the collection of points (x,y,z) (or points (x,y) in case of $f(x,y)$) where the value of the function is a constant.

e.g. the level curves of the function $f(x,y) = x^2+y^2$ are circles in the xy plane, centered at the origin.

- Tangent line to a level curve of a function $f(x,y)$ at point $P(x_0,y_0)$

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$$

Example: $f(x,y) = x^2 + y^2$, and look at the level curve $f(x,y) = 25$, i.e. circle centered at origin w/ radius 5.



$f_x = 2x$; $f_y = 2y$, so $f_x(x_0, y_0) = 2x_0$

$f_y(x_0, y_0) = 2y_0$

Therefore, eq^n of tangent line at (x_0, y_0) :

$$2x_0 (x - x_0) + 2y_0 (y - y_0) = 0$$

i.e. $\boxed{x_0 (x - x_0) + y_0 (y - y_0) = 0}$

Let's use this to find the equation of the tangent line to the circle at different points:

① at $(5,0)$: $5(x-5) + 0(y-0) = 0$

i.e. $\boxed{x = 5}$

② at $\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$: $\frac{5}{\sqrt{2}}\left(x - \frac{5}{\sqrt{2}}\right) + \frac{5}{\sqrt{2}}\left(y - \frac{5}{\sqrt{2}}\right) = 0$

i.e. $x - \frac{5}{\sqrt{2}} + y - \frac{5}{\sqrt{2}} = 0$

i.e. $x + y = \frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}} = \frac{\overset{\sqrt{2}}{\cancel{2}} \times 5}{\cancel{\sqrt{2}}} = 5\sqrt{2}$

i.e. $\boxed{x + y = 5\sqrt{2}}$

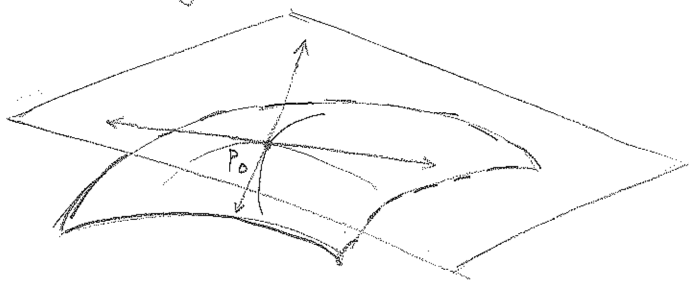
③ at $(0,5)$: $0(x-0) + 5(y-5) = 0$

i.e. $\boxed{y = 5}$

Okay, this was a function $f(x,y)$ of two variables, and we talked about the tangent lines to its level curves.

- Now lets go one dimension up and look at functions of three variables $f(x,y,z)$. Here we talk about <u>level surfaces</u> and <u>tangent planes</u>.

- <u>Conceptually</u>, the tangent plane to a surface at a point $P_0(x_0,y_0,z_0)$ is the plane that is composed of all the tangent lines at P_0 to curves that lie in the surface and pass through P_0.



- <u>Practically</u>, we define it like this:

The tangent plane to the level surface $f(x,y,z)=c$ at the point $P_0(x_0,y_0,z_0)$ is the plane through the point P_0, with normal vector $(\nabla f)_{P_0}$.

The <u>normal line</u> of the surface at P_0 is the line through P_0 which is parallel to $(\nabla f)_{P_0}$.

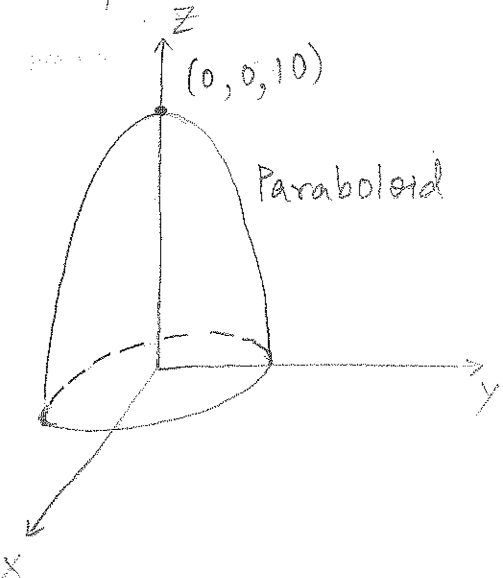
The tangent plane is given by the equation

$$\boxed{f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0}$$

the normal line is given by:

$$\boxed{\begin{aligned} x &= x_0 + f_x(x_0,y_0,z_0)\,t \\ y &= y_0 + f_y(x_0,y_0,z_0)\,t \\ z &= z_0 + f_z(x_0,y_0,z_0)\,t \end{aligned}}$$

Example Find the tangent plane and normal line to the level surface $f(x,y,z) = x^2+y^2+z-10=0$ at the point $P(2,1,5)$



Solution $f_x = 2x$, $f_y = 2y$, $f_z = 1$

$f_x(2,1,5) = 2\times 2 = 4$

$f_y(2,1,5) = 2\times 1 = 2$

$f_z(2,1,5) = 1$

So, $\nabla f|_P = 4\hat{i} + 2\hat{j} + \hat{k}$

and the tangent plane is:

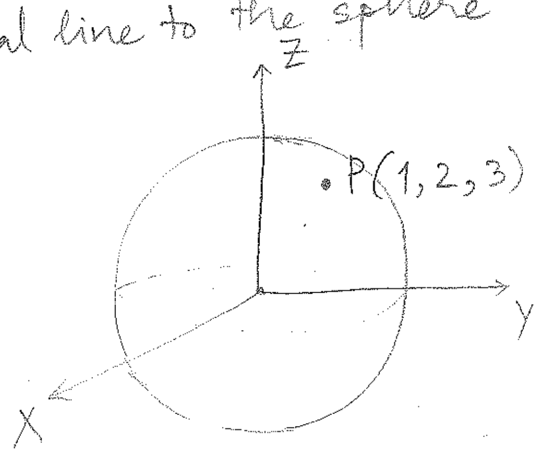
$$4(x-2)+2(y-1)+1(z-5)=0$$

i.e. $\boxed{4x+2y+z=15}$

and the normal line is:

$$\boxed{\begin{aligned} x &= 2 + 4t \\ y &= 1 + 2t \\ z &= 5 + t \end{aligned}}$$

Example. Find the tangent plane and normal line to the sphere $x^2+y^2+z^2=14$ at the point $P(1,2,3)$



Solution. First we write it as the level surface of a function, let $f(x,y,z) = x^2+y^2+z^2$ then the sphere is the level surface

$f(x,y,z) = 14$

$\nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

so $\nabla f\Big|_{(1,2,3)} = 2\hat{i} + 4\hat{j} + 6\hat{k}$

the tangent plane is:

$$\boxed{2(x-1) + 4(y-2) + 6(z-3) = 0}$$

and the normal line is:

$$\boxed{\begin{aligned} x &= 1 + 2t \\ y &= 2 + 4t \\ z &= 3 + 6t \end{aligned}}$$

Example. Find the tangent plane to the surface $z = e^{xy} + \cos y$ at the point $(0, 0, 2)$

Solution: let $f(x,y,z) = e^{xy} + \cos y - z$, then the given surface is the level surface $f(x,y,z) = 0$

$$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k} = ye^{xy}\hat{i} + (xe^{xy} - \sin y)\hat{j} - \hat{k}$$

so $\nabla f\Big|_{(0,0,2)} = 0e^0\hat{i} + (0e^0 - \sin 0)\hat{j} - \hat{k} = -\hat{k}$

so, the tangent plane is:

$$0(x-0) + 0(y-0) - 1(z-2) = 0$$

i.e. $\boxed{z = 2}$