Lecture 9   Wednesday 6/8

14.7 Extreme values and saddle points

- Recall from functions of one variable:

→ If $f(x)$ has a local maximum or local minimum at $x=a$ then $f'(a)=0$ $\left(\frac{df}{dx}\Big|_{x=a}=0\right)$.

→ If we know that $f'(a)=0$ and $f''(a)<0$ then a is a point of local maximum.

→ If we know that $f'(a)=0$ and $f''(a)>0$ then a is a point of local minimum.

- Similarly we can talk about local max / min of functions of several variables.
- Note the difference b/w local max / min and global max / min.
- We say the function $f(x,y)$ has a local max (min) at the point (a,b) if there is a small disk around (a,b) in the xy plane so that $f(a,b) > f(x,y)$ $\big(f(a,b) < f(x,y)\big)$ for all (x,y) in that disk.
- First derivative test for local max / min:

  If (a,b) is a point of local max or local min of a function $f(x,y)$ then $\boxed{f_x(a,b) = f_y(a,b) = 0}$.

- Definition: If (a,b) is a point satisfying $f_x(a,b) = f_y(a,b) = 0$ then (a,b) is called a critical point of the function f.

Example: Find the critical points of the function

$$f(x,y) = x^3 + 3xy - y^2$$

Solution: $f_x = 3x^2 + 3y$; $f_y = 3x - 2y$

critical point $\Rightarrow$ $f_x = 0$, $f_y = 0$

i.e. $3x^2 + 3y = 0$; $3x - 2y = 0$

(we need to solve this pair of equations in x and y, to find the critical points.)

$\Rightarrow$ $x^2 + y = 0$ $\Rightarrow$ $\boxed{y = -x^2}$

substitute this in the second equation.

$$3x - 2y = 0 \Rightarrow 3x - 2(-x^2) = 0$$

$$\Rightarrow 3x + 2x^2 = 0$$

$$\Rightarrow x(3 + 2x) = 0$$

$$\Rightarrow \boxed{x = 0} \quad \text{or} \quad 3 + 2x = 0$$

$$\Rightarrow \boxed{x = -\frac{3}{2}}$$

If $x = 0$, then

$y = -0^2 = 0$

so $\boxed{(0,0)}$ is one critical point.

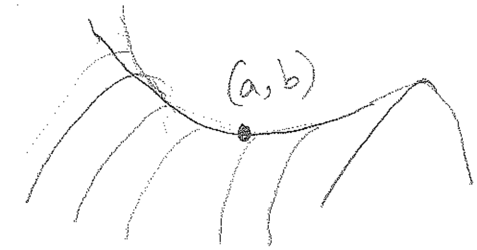
If $x = -\frac{3}{2}$ then

$y = -\left(-\frac{3}{2}\right)^2 = -\frac{9}{4}$

so $\boxed{\left(-\frac{3}{2}, -\frac{9}{4}\right)}$ is another critical point.

- Just being a critical point does not guarantee that it is a point of local max or local min. It could also be a saddle point. (a,b) is said to be a saddle point of f if for any disk containing (a,b), there are points (x,y) in the disk so that $f(x,y) > f(a,b)$ and also points (x,y) in the disk so that $f(x,y) < f(a,b)$.

The graph of the function f then looks like a saddle, hence the name <u>saddle point</u>.



- Just like for functions of one variable, we have a second derivative test to decide if a critical point is a point of local max or min.

$$\boxed{D = f_{xx} f_{yy} - f_{xy}^2}$$ is called the <u>discriminant</u> of f

Then, the second derivative test says:

1. If $D|_{(a,b)} > 0$ and $f_{xx}|_{(a,b)} < 0$ then (a,b) is a point of local max.

2. If $D|_{(a,b)} > 0$ and $f_{xx}|_{(a,b)} > 0$ then (a,b) is a point of local min.

3. If $D|_{(a,b)} < 0$ then (a,b) is a saddle point

4. If $D|_{(a,b)} = 0$ then we can't conclude anything.

<u>Example</u>. For the function $f(x,y) = x^3 - xy + y^3$, find its critical points and for each, say whether it is a local max, local min or saddle point.

<u>Solution</u> $f_x = 3x^2 - y$; $f_y = -x + 3y^2$

critical point $\Rightarrow$ $f_x = 0$, $f_y = 0$

i.e. $3x^2 - y = 0$, $-x + 3y^2 = 0$

$\Rightarrow$ $y = 3x^2$. substitute in get: $-x + 3(3x^2)^2 = 0$

$\Rightarrow -x + 27x^4 = 0$

3 $\Rightarrow x(-1 + 27x^3) = 0$

$\Rightarrow \boxed{x=0}$ or $\underbrace{-1+27x^3=0}$

$\Rightarrow y = 3\times 0^2 = 0$

so $(0,0)$ is a critical point.

$\Rightarrow 27x^3 = 1$

$\Rightarrow x^3 = \frac{1}{27} \Rightarrow x = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$

$\boxed{x = \frac{1}{3}}$

$\Rightarrow y = 3\times\left(\frac{1}{3}\right)^2 = \frac{1}{3}$

so $\left(\frac{1}{3}, \frac{1}{3}\right)$ is a critical point.

Also $f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = -1$

so $D = f_{xx}f_{yy} - f_{xy}^2$

$= 6x \times 6y - (-1)^2$

$\boxed{D = 36xy - 1}$

at $(0,0)$: $D\big|_{(0,0)} = 36\times 0\times 0 - 1 = -1 < 0 \Rightarrow (0,0)$ is a saddle point

at $\left(\frac{1}{3},\frac{1}{3}\right)$: $D\big|_{\left(\frac{1}{3},\frac{1}{3}\right)} = 36\times\frac{1}{3}\times\frac{1}{3} - 1 = 4-1 = 3 > 0$

and $f_{xx}\big|_{\left(\frac{1}{3},\frac{1}{3}\right)} = 6\times\frac{1}{3} = 2 > 0$

$\Rightarrow \left(\frac{1}{3},\frac{1}{3}\right)$ is a point of local minimum.

14.8 Lagrange multipliers

- Next we will talk about the problem of finding local max / min of functions $f(x,y,z)$ subject to a constraint $g(x,y,z)=0$.
- This means we're trying to find the points on the surface $g(x,y,z)=0$, where the function f attains its maximum / minimum value (among points on this surface).
- We use the method of Lagrange multipliers, which says that the points of max / min satisfy the equation $\boxed{\nabla f = \lambda \nabla g}$ for some number λ. So we need to solve this equation to get points (x,y,z).

Example Find the maximum volume of a rectangular box given that the sum of its length, width and height is 60 m.

Solution. Lets denote length by the variable x
width by the variable y
height by the variable z

Then, we're trying to find the maximum value of the function

$f(x,y,z) = \text{volume} = xyz$

subject to the constraint $x+y+z = 60$

i.e., setting $g(x,y,z) = x+y+z-60$, this is the constraint $g(x,y,z) = 0$.

$\nabla f = \langle yz, xz, xy \rangle$; $\nabla g = \langle 1,1,1 \rangle$

$\nabla f = \lambda \nabla g$ is :

$$\langle yz, xz, xy \rangle = \lambda \langle 1,1,1 \rangle = \langle \lambda, \lambda, \lambda \rangle$$

i.e. $yz = \lambda$ — (1)
$xz = \lambda$ — (2)
$xy = \lambda$ — (3)

Equation (1) implies $y = \frac{\lambda}{z}$. Equation (2) implies $x = \frac{\lambda}{z}$

Putting the two together implies $\boxed{y = x}$

Equation (1) implies $z = \frac{\lambda}{y}$. Equation (3) implies $x = \frac{\lambda}{y}$

Therefore $\boxed{z = x}$.

Plugging in $y = x$ and $z = x$ into the constraint $g(x,y,z) = 0$:

$x+y+z = 60$

$\Rightarrow x+x+x = 60 \Rightarrow 3x = 60 \Rightarrow \boxed{x = 20} \Rightarrow \boxed{y = 20}$
$\boxed{z = 20}$

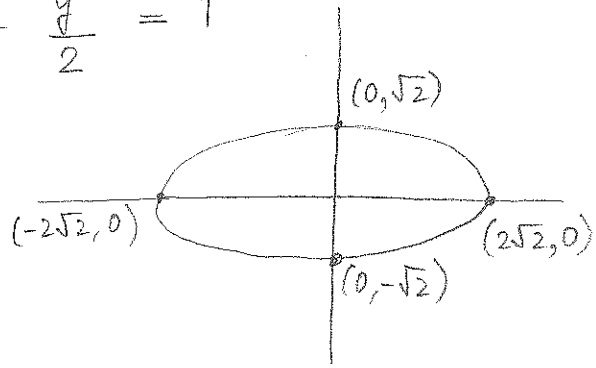
Therefore the maximum value is attained for $(x, y, z) = (20, 20, 20)$

i.e. the maximum possible volume of the box is

$$f(20,20,20) = 20 \times 20 \times 20 = \boxed{8000\, m^3}$$

Example: Find the max and min values of the function

$f(x,y) = xy$ on the ellipse $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1$



Solution. Let $g(x,y) = \dfrac{x^2}{8} + \dfrac{y^2}{2} - 1$

$\nabla f = \langle y, x \rangle$

$\nabla g = \langle \frac{x}{4}, y \rangle$

By the method of Lagrange multipliers

$$\nabla f = \lambda \nabla g$$

$$\langle y, x \rangle = \lambda \langle \tfrac{x}{4}, y \rangle$$

$$\Rightarrow \quad y = \lambda \cdot \frac{x}{4} \quad \text{—— ①}$$

$$x = \lambda y \quad \text{—— ②}$$

divide ① by ②: $\dfrac{y}{x} = \dfrac{\lambda \frac{x}{4}}{\lambda y} = \dfrac{x}{4y}$

$$\Rightarrow \quad 4y^2 = x^2$$

substitute in constraint equation

$$\frac{4y^2}{8} + \frac{y^2}{2} = 1 \Rightarrow \frac{y^2}{2} + \frac{y^2}{2} = 1 \Rightarrow y^2 = 1$$

$$\Rightarrow \boxed{y = \pm 1}$$

if $y = 1$, $x^2 = 4 \times (1)^2 = 4$ so $x = \pm 2$

so $(2,1)$ and $(-2,1)$ are candidates for max/min

if $y = -1$, $x^2 = 4 \times (-1)^2 = 4$ so $x = \pm 2$

so $(2,-1)$ and $(-2,-1)$ are also candidates for max min

$f(2,1) = 2$ $\qquad$ $f(2,-1) = -2$

$f(-2,1) = -2$ $\qquad$ $f(-2,-1) = 2$

since the points of max/min must be amongst the points $(2,1)$, $(-2,1)$, $(2,-1)$ and $(-2,-1)$, we conclude that the max value is 2, and is attained at points $(2,1)$ and $(-2,-1)$ while the min value is -2, and is attained at the points $(-2,1)$ and $(2,-1)$.