Math 114 Quiz 4

Thu, 6/9

Name :

1. Let $f(x,y) = x^2y^3 + \ln(xy)$, and answer the following questions for the derivatives at the point $(1,1)$.
   (a) What is the derivative of f in the direction $\hat{\mathbf{i}} - \hat{\mathbf{j}}$?
   (b) Find a direction $\vec{u} = \langle u_1, u_2 \rangle$ in which the directional derivative equals 4.
   (c) (Extra credit. Attempt this part only if you have completed the first two parts!) Is there a direction in which the derivative is 6? Explain your answer.

Solution: (a) $f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(x^2y^3 + \ln(xy)\right) = 2xy^3 + \dfrac{y}{xy} = 2xy^3 + \dfrac{1}{x}$

$f_y = \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left(x^2y^3 + \ln(xy)\right) = 3x^2y^2 + \dfrac{x}{xy} = 3x^2y^2 + \dfrac{1}{y}$

so $\nabla f = \left(2xy^3 + \dfrac{1}{x}\right)\hat{i} + \left(3x^2y^2 + \dfrac{1}{y}\right)\hat{j}$

$\nabla f\big|_{(1,1)} = \left(2\times 1\times 1^3 + \dfrac{1}{1}\right)\hat{i} + \left(3\times 1^2\times 1^2 + \dfrac{1}{1}\right)\hat{j}$

$= 3\hat{i} + 4\hat{j}$

unit vector in direction of $\hat{i} - \hat{j}$ is: $\vec{u} = \dfrac{\hat{i} - \hat{j}}{|\hat{i} - \hat{j}|} = \dfrac{\hat{i} - \hat{j}}{\sqrt{1^2 + (-1)^2}}$

$= \dfrac{\hat{i} - \hat{j}}{\sqrt{2}} = \dfrac{\hat{i}}{\sqrt{2}} - \dfrac{\hat{j}}{\sqrt{2}}$

Derivative of f in direction $\hat{i} - \hat{j}$ is:

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = (3\hat{i} + 4\hat{j}) \cdot \left(\dfrac{\hat{i}}{\sqrt{2}} - \dfrac{\hat{j}}{\sqrt{2}}\right) = \dfrac{3}{\sqrt{2}} - \dfrac{4}{\sqrt{2}} = \boxed{\dfrac{-1}{\sqrt{2}}}$

(b) suppose $\vec{u} = \langle u_1, u_2 \rangle$ is a unit vector so that the directional derivative in direction of $\vec{u}$ equals 4.

i.e. $D_{\vec{u}} f = \nabla f \cdot \vec{u} = 4$

i.e. $\langle 3, 4 \rangle \cdot \langle u_1, u_2 \rangle = 4$

i.e. $3u_1 + 4u_2 = 4$ must be satisfied.

$\vec{u} = \langle 0, 1 \rangle$ is a unit vector that satisfies this.

(c) The maximum value of the directional derivative occurs when we take $\vec{u}$ to be the unit vector in direction of ∇f.

i.e. when $\vec{u} = \dfrac{3\hat{i} + 4\hat{j}}{\sqrt{3^2 + 4^2}} = \dfrac{3\hat{i} + 4\hat{j}}{5} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$

so the maximum possible value of the directional derivative is:

$$\nabla f \cdot \vec{u} = (3\hat{i} + 4\hat{j}) \cdot \left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right)$$

$$= 3 \times \frac{3}{5} + 4 \times \frac{4}{5} = \frac{9}{5} + \frac{16}{5} = \frac{25}{5} = \boxed{5}$$

So 5 is the maximum possible value of the directional derivative. In other words the value can never be bigger than 5. In particular, since $6 > 5$, it can never have the value 6.