Math 114 Quiz 7

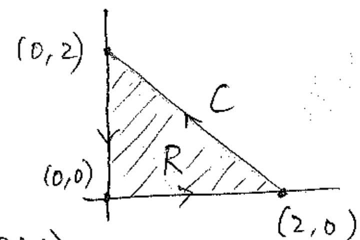
Mon, 6/27

Name :

1. Use Green's theorem to evaluate the integral

$$\oint_C ((\ln(x^3+1)+y)dx + (3x + e^{\sin y})dy)$$

where C is the triangle bounded by $x = 0$, $y = 0$, and $x + y = 2$.



Solution: $M = \ln(x^3+1) + y$

$N = 3x + e^{\sin y}$

Green's theorem $\Rightarrow$ $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(3x + e^{\sin y}\right) = 3$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\ln(x^3+1) + y\right) = 1$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3 - 1 = 2$$

$$= \iint_R 2\,dA = 2\iint_R dA = 2 \times \text{area of } R$$

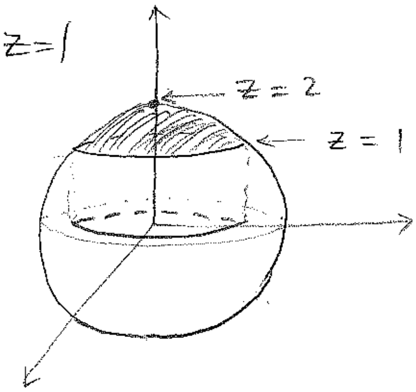
$$= 2 \times \left(\frac{1}{2} \times 2 \times 2\right)$$

$$= \boxed{4}$$

2. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 4$ that lies between the planes $z = 1$ and $z = 2$

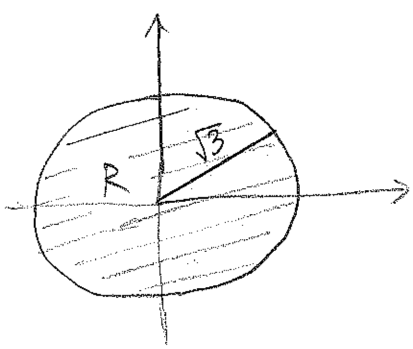
Solution. the sphere intersects the plane $z=1$ in a circle: $x^2+y^2+1^2=4$, $z=1$

i.e. $x^2+y^2=3$, $z=1$



therefore shadow region in x-y plane is:

$x^2+y^2 \leq 3$.



$$f(x,y,z) = x^2+y^2+z^2-1$$

$$\vec{p} = \hat{k}$$

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$|\nabla f| = \sqrt{4x^2+4y^2+4z^2}$$

$$= \sqrt{4(\underline{x^2+y^2+z^2})} = \sqrt{4 \times \underline{4}} = 4$$

(on the surface of the sphere, $x^2+y^2+z^2=4$)

$$|\nabla f \cdot \vec{p}| = |2z| = 2z = 2\sqrt{4-x^2-y^2}$$

$$\therefore \text{Area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot p|} = \iint_R \frac{4}{2\sqrt{4-x^2-y^2}}\,dx\,dy = \iint_R \frac{2}{\sqrt{4-x^2-y^2}} = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{3}} \frac{r\,dr\,d\theta}{\sqrt{4-r^2}}$$

let $u = 4-r^2$, so $du = -2r\,dr$, $r\,dr = -\frac{1}{2}du$. $r=0 \Rightarrow u=4$, $r=\sqrt{3} \Rightarrow u=1$

$$\therefore \int_{r=0}^{\sqrt{3}} \frac{r\,dr}{\sqrt{4-r^2}} = \frac{-1}{2}\int_4^1 \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_1^4 u^{-1/2}\,du = \frac{1}{2} \times 2u^{1/2}\Big|_1^4 = (4^{1/2} - 1^{1/2}) = 2-1 = 1$$

$$\Rightarrow = 2\int_0^{2\pi} 1\,d\theta = 2 \times 2\pi = \boxed{4\pi}$$

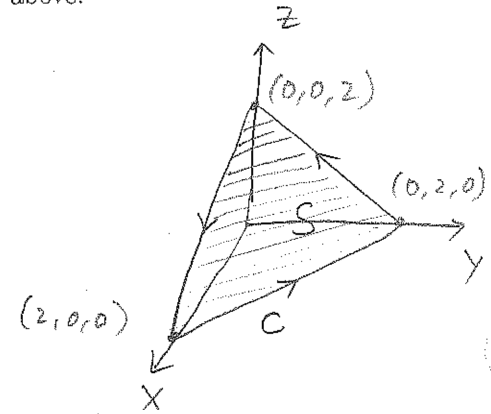
3. Use Stokes' theorem to evaluate the line integral of

$$\vec{F} = (y - 2z)\hat{\text{i}} + (z - 2x)\hat{\text{j}} + (x - 2y)\hat{\text{k}}$$

around the curve C, where C is the boundary of the portion of the plane

$$x + y + z = 2$$

in the first octant, traversed counterclockwise when viewed from above.



<u>Solution</u>: By Stokes' theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, d\sigma$$

we will evaluate the line integral by computing the surface integral on the right.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-2z & z-2x & x-2y \end{vmatrix} = \hat{i}(-2-1) - \hat{j}(1-(-2)) + \hat{k}(-2-1) = -3\hat{i} - 3\hat{j} - 3\hat{k}$$

normal vector of the plane $= \hat{i} + \hat{j} + \hat{k}$

unit normal vector $\vec{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{1^2+1^2+1^2}} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

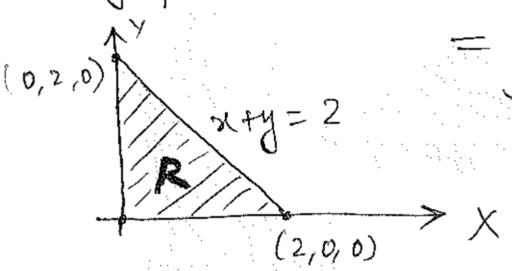
so $(\vec{\nabla}\times\vec{F})\cdot\vec{n} = \dfrac{-3-3-3}{\sqrt{3}} = \dfrac{-9}{\sqrt{3}} = -3\sqrt{3}$

so, $\displaystyle\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, d\sigma = \iint_S (-3\sqrt{3})\, d\sigma$

$$f(x,y,z) = x+y+z-2$$
$$|\nabla f| = |\hat{i}+\hat{j}+\hat{k}| = \sqrt{3}$$
$$\vec{p} = \hat{k}$$
$$|\vec{\nabla} f\cdot\vec{p}| = 1$$

Shadow region in x-y plane:

$$= \iint_R (-3\sqrt{3})\frac{|\nabla f|}{|\nabla f\cdot \vec{p}|}\, dx\, dy$$



$$= \iint_R (-3\sqrt{3})\cdot\frac{\sqrt{3}}{1}\, dx\, dy = -9\iint_R dx\, dy = -9\times \text{area of } R$$

$$= -9\times\frac{1}{2}\times 2\times 2$$

$$= \boxed{-18}$$