Math 601 Homework 1 Solutions to selected problems

1. **Problem 6.** A space X is said to be contractible if the identity map $1_X : X \to X$ is homotopic to a constant map.
(a) Show that any convex open set in $\mathbb{R}^n$ is contractible.
(b) Show that a contractible space is path connected.
(c) Show that if Y is contractible, then all maps of $X \to Y$ are homotopic to one another.
(d) Show that if X is contractible and Y is path connected, then all maps of $X \to Y$ are homotopic.

Solution:
(a)Let $X \subset \mathbb{R}^n$ be convex, and let $c \in X$. Let $f : X \to X$ be the constant map $f(x) = c$. Then $H : X \times I \to X$ defined by:

$$H(x,t) = tc + (1-t)x$$

is a homotopy between 1_X (at t=0) and f (at t=1). Note that the map makes sense as $H(x,t) \in X$ for every $x \in X$ since X is convex.

(b)Let X be contractible, i.e. there's a homotopy H s.t. $H(x,0) = x$ and $H(x,1) = c$ for every $x \in X$, and for some fixed $c \in X$. Then, for each $x \in X$, $f_x(t) = H(x,t)$ is a path between x and c in X. So every point of X is connected to the fixed point c, by a path. Therefore any two points x_1 and x_2 of X can be joined via a path through c.

(c)Let Y be contractible, i.e. there is a homotopy H between 1_Y and a constant map $f(y) = c$. Let $g : X \to Y$. Then $H \circ g : X \times I$ is a homotopy between g and the constant map $\tilde{f} : X \to Y$, $\tilde{f}(x) = c$ for each x. So every map $g : X \to Y$ is homotopic to the constant map $\tilde{f}$. Since homotopy is an equivalence relation, this implies all maps $X \to Y$ are homotopic.

(d)X is contractible, i.e. there's a homotopy H as in (b). Let $f : X \to Y$. Then $f \circ H : X \times I \to Y$ is a homotopy between f and the constant map $\tilde{f} \equiv f(c)$. Now, let $f_1 \equiv y_1$ and $f_2 \equiv y_2$ be any two constant maps $X \to Y$. Let $\alpha : I \to Y$ be a path between y_1 and y_2. Then $F(x,t) = \alpha(t)$ is a homotopy between f_1 and f_2. So we have :
1)Every map $X \to Y$ is homotopic to **some** constant map.
2)Any two constant maps $X \to Y$ are homotopic.
Since homotopy is an equivalence relation, these two facts together imply that any two maps $X \to Y$ are homotopic.

2. **Problem 8.** Show that the operation $*$ of composition of paths induces a corresponding well-defined operation $*$ of composition of fixed end point homotopy classes of paths, that is associative, has left and right identities, and has inverses.

   Solution: The operation on fixed endpoint homotopy classes of paths in a space X is defined as :

$$[\alpha] * [\beta] = [\alpha * \beta]$$

   Well-defined: Suppose α_1 homotopic to α_2 and β_1 homotopic to β_2 via $F, G : I \times I \to X$ respectively. Then $H : I \times I \to X$ given by

$$H(x,t) = \begin{cases} F(2t, s) & \text{if } 0 \le t \le \frac{1}{2} \\ G(2t-1, s) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

   is a homotopy between $\alpha_1 * \beta_1$ and $\alpha_2 * \beta_2$. Therefore $[\alpha_1 * \beta_1] = [\alpha_2 * \beta_2]$ and the operation $*$ is well-defined on homotopy classes of paths in X.
   Associative : We need to show that for paths f, g, h with contiguous images, the path $(f * g) * h$ is homotopic to the path $f * (g * h)$. Note that

$$\big((f*g)*h\big)(s) = \begin{cases} f(4s) & \text{if } 0 \le s \le \frac{1}{4} \\ g(4s-1) & \text{if } \frac{1}{4} \le s \le \frac{1}{2} \\ h(2s-1) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

   and also

$$\big(f*(g*h)\big)(s) = \begin{cases} f(2s) & \text{if } 0 \le s \le \frac{1}{2} \\ g(4s-2) & \text{if } \frac{1}{2} \le s \le \frac{3}{4} \\ h(4s-3) & \text{if } \frac{3}{4} \le s \le 1 \end{cases}$$

   Then $H : I \times I \to X$ defined by

$$H(t,s) = \begin{cases} f(\frac{4s}{1+t}) & \text{if } 0 \le s \le \frac{1+t}{4} \\ g(4s-1-t) & \text{if } \frac{1+t}{4} \le s \le \frac{2+t}{4} \\ h(\frac{4s}{2-t} - \frac{2+t}{2-t}) & \text{if } \frac{2+t}{4} \le s \le 1 \end{cases}$$

   is a homotopy between $(f * g) * h$ and $f * (g * h)$. That H is continuous can be seen by the pasting lemma.
   Identities : For homotopy classes of paths between points x and y in X, we claim that the constant maps $c_1(s) = p$ and $c_2(s) = q$ are left and right identities of the $*$ operation respectively. To see this, construct an explicit homotopy between f and $c_1 * f$ or between f and $f * c_2$ as done above for showing associativity, or use the idea of reparametrization of paths described in Hatcher, page 27.
   Inverses : For a path f, the path f^{-1} defined by $f^{-1}(s) = f(1-s)$ is an inverse for f with respect to the operation $*$. Again, to check that $f * f^{-1}$ is homotopic to c_1 or that $f^{-1} * f$ is homotopic to c_2, explicitly construct homotopies or use the idea of reparametrization of paths described in Hatcher.

3. **Problem 12.** Show that a continuous map $f : (X, x_0) \to (Y, y_0)$ induces a group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Solution: Define the map f_* as :

$$f_*[\alpha] = [f \circ \alpha]$$

for α a representative of a homotopy class of loops.
Well-defined : If α is homotopic to β via $H : I \times I \to X$ then $f \circ \alpha$ is homotopic to $f \circ \beta$ via $f \circ H$.
f_* is a group homomorphism : We need to show that

$$f_*\big([\alpha] * [\gamma]\big) = f_*[\alpha] * f_*[\gamma] \tag{1}$$

Recall that

$$\big([\alpha] * [\gamma]\big)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

So

$$f_*\big([\alpha] * [\gamma]\big)(t) = \begin{cases} f \circ \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ f \circ \gamma(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

But this is the same as $[f \circ \alpha] * [f \circ \gamma]$, which by the definition of f_*, is the right hand side of eqn (1).