Math 601 Homework 2 Solutions to selected problems

1. **Problem 1.** Show that the map p: $\mathbb{R}^1 \to S^1$ defined by

$$p(x) = (cos2\pi x, sin2\pi x)$$

is a covering map.

Solution:
For each point $q_0 \in S^1$, $q_0 = (cos2\pi x_0, sin2\pi x_0)$ for some $x_0 \in [0,1)$. Let U be an open neighbourhood of q_0, $U = \{(cos2\pi x, sin2\pi x) : x \in (x_0 - \delta, x_0 + \delta)\}$, for a fixed $0 < \delta < \frac{1}{2}$. Then $p^{-1}(U) = \sqcup_{n\in\mathbb{Z}}(n + x_0 - \delta, n + x_0 + \delta)$. Restricted to each $U_n = (n + x_0 - \delta, n + x_0 + \delta)$, p is a homeomorphism onto U. Therefore U is an evenly covered neighbourhood of q_0. We can get such a set U for every point $q_0 \in S^1$, so p is a covering map.$\square$

2. **Problem 6.** Let $p : E \to B$ be a covering map, and suppose that B is connected. Show that if $p^{-1}(b)$ has k elements for some $b \in B$, then it has k elements for every $b \in B$. In this case, we say that E is a k-fold covering of B.

Solution:
Let $B_n = \{b \in B|\ p^{-1}(b)$ has n elements$\}$. Then $B = \sqcup_n B_n$. We're given that $B_k \neq \phi$. It's enough to show that each B_n is an open set, because the fact that B is connected would then imply that all but one of the B_n's is empty.
Let $b \in B_n$, and let $U \subseteq B$ be an evenly covered neighbourhood of b, $p^{-1}(U) = \sqcup_j U_j$ such that p restricted to each U_j is a homeomorphism onto U. Then, since $p^{-1}(b)$ has n elements, it implies that the number of sets U_j is n. Thus for every $x \in U$, its preimages lie in the U_j's, and hence $p^{-1}(x)$ has n elements for each $x \in U$. Therefore $U \subseteq B_n$, so B_n is open.$\square$

3. **Problem 12.** Let $f : S^1 \to S^1$ be defined by $f(z) = z^n$. Note that $f(1) = 1$. Compute the induced homomorphism $f* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$.

Solution:
First observe that $\pi_1(S^1, 1)$ is isomorphic to the additive group $\mathbb{Z}$. To see this one can define a function $\phi : \pi_1(S^1, 1) \to \mathbb{Z}$ by defining it on loops as follows. Let $p : \mathbb{R} \to S^1$ be the covering map from Problem 1. For a loop $\alpha : I \to S^1$, let $\tilde{\alpha} : I \to \mathbb{R}$ be its lift with initial point $0 \in \mathbb{R}$. Then define $\phi([\alpha]) = \tilde{\alpha}(1)$. (Checking that this map is well-defined and is actually a group isomorphism, was the content of Problem 10.)

So, $f* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is actually a group homomorphism F from $\mathbb{Z}$ to $\mathbb{Z}$, and this is described by specifying the image of the generator $1 \in \mathbb{Z}$ under F. Therefore consider the loop $\gamma(t) = e^{2\pi it}$, which is a generator of $\pi_1(S^1, 1)$. (It corresponds to $1 \in \mathbb{Z}$ under the isomorphism ϕ.) Under the map f_* it gets sent to the loop $f \circ \gamma(t) = (e^{2\pi it})^n = e^{2\pi int}$. The lift of this path starting at $0 \in \mathbb{R}$ is the path $\tilde{\gamma} : I \to \mathbb{R}$, $\tilde{\gamma}(t) = nt$. Observe that $\tilde{\gamma}(1) = n$, so by the correspondence ϕ, this implies that $f \circ \gamma$, i.e., $f_*[\gamma]$, corresponds to n times the generator in $\pi_1(S^1, 1)$.

We conclude that the homomorphism $f* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is the homomorphism $\mathbb{Z} \to \mathbb{Z}$ that sends 1 to n. $\square$

Note : If you want to do the proof without going to the universal cover $\mathbb{R}$, you will need to show (by indicating an explicit homotopy) that the loop $f \circ \gamma(t) = e^{2\pi int}$ is homotopic to the loop $\underbrace{\gamma * \cdots * \gamma}_{n\text{-times}}$, where $*$ means concatenation of loops. This claim needs a proof!

4. **(Hatcher, page 38, Problem 9)** Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Use the Borsuk-Ulam theorem to show that there is one plane P in $\mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.

   Solution : We're going to define a map $S^3 \to \mathbb{R}^3$ on which we'll use the conclusion of the Borsuk-Ulam theorem.

   For this, we claim that every point on S^3 is in one-to-one correspondence with an oriented two-plane in $\mathbb{R}^3$. Any point $(a, b, c, d) \in S^3 - \{(0, 0, 0, \pm 1)\}$ corresponds to a two-plane $P : ax + by + cz + d = 0$. And given a two-plane $P : ax + by + cz + d = 0$, we have the corresponding point $\frac{1}{\sqrt{a^2+b^2+c^2+d^2}}(a, b, c, d) \in S^3 - \{(0, 0, 0, \pm 1)\}$.

   Let's also associate to the two-plane $ax + by + cz + d = 0$, the direction vector $\mathbf{v} = (a, b, c) \in \mathbb{R}^3$. Observe that P divides $\mathbb{R}^3$ into two half-spaces. Translating $\mathbf{v}$ so its base point lies on the plane P, we denote by A the half-space that the endpoint of $\mathbf{v}$ lies in.

   Now, define a function $f : S^3 \to \mathbb{R}^3$, as :

$$f(a, b, c, d) = (m(A_1 \cap A), m(A_1 \cap A), m(A_1 \cap A)),$$

   where m is the Lebesgue measure on $\mathbb{R}^3$. The function f is certainly continuous on $S^3 - \{(0, 0, 0, \pm 1)\}$, and can be extended to a continuous function on all of S^3, as follows.

   As (a, b, c, d) tends to $(0, 0, 0, 1)$, the plane P moves away from the origin, but its direction vector (a, b, c) points towards the origin. Since all the A_i's are compact, eventually, for the plane P being far enough out (i.e. (a, b, c, d) near enough to $(0, 0, 0, 1)$), each A_i is completely contained in the half-space A. Therefore eventually $m(A_i \cap A) = m(A_i)$ for each i, for all (a, b, c, d) near enough to $(0, 0, 0, 1)$. So we can define $f(0, 0, 0, 1) = (m(A_1), m(A_2), m(A_3))$. Similar analysis shows we can define $f(0, 0, 0, -1) = 0$, since all A_i's lie in the complement of A, for (a, b, c, d) near enough to $(0, 0, 0, -1)$.

Now that we've got the continuous function $S^3 \to \mathbb{R}^3$, the Borsuk-Ulam theorem tells us that there's a pair of antipodal points $\{x, -x\}$ in S^3 with $f(x) = f(-x)$. But $f(x)$, $f(-x)$ respectively give the measures of the portions of each A_i that lie in the two half-spaces of the two-plane P corresponding to x. So $f(x) = f(-x)$ means that the plane P divides each A_i into two sets of equal measure.$\square$