Math 601 Homework 3 Solutions to selected problems

1. **Problem 5.** Show that the fundamental group of $R^3 - \{0\}$ is trivial, i.e., show that this space is simply connected.
   Hint: It may be helpful to approximate a loop in $\mathbb{R}^3 - \{0\}$ by a polygonal loop.

   Solution:
   The map $F : \mathbb{R}^3 - \{0\} \to S^2$ given by $x \mapsto \frac{x}{|x|}$ is a deformation retraction of $\mathbb{R}^3 - \{0\}$ onto the 2-sphere. Therefore $\pi_1(\mathbb{R}^3 - \{0\}) \cong \pi_1(S^2)$. So we just need to show that S^2 is simply connected.

   Given any loop $\gamma \to S^2$, we want to show that γ is homotopic to the trivial loop. If there exists a point $p \in S^2$ not in the image of γ, then the image of γ is contained in $S^2 - \{x\}$, which is homeomorphic to $\mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, every loop in it is homotopic to the trivial loop. Hence γ is homotopic to the trivial loop in S^2.

   Note that γ is merely continuous and not smooth, and continuous functions can be pretty badly-behaved (e.g. space-filling curves). If there's no $p \in S^2$ that's **not** in the image of γ (i.e. if γ is surjective) then we need to first homotope γ to a curve that's not surjective, and then apply the argument in the previous paragraph.

   Exercise : Give a rigorous argument explaining how, given a point $p \in S^2$ you will homotope the curve γ to a curve that doesn't intersect p.

   Note : This argument works for all S^{m-1}, $m \geq 3$ as well, so we actually get that for all $m \geq 3$, $\pi_1(\mathbb{R}^m - \{0\}) \cong \pi_1(S^{m-1}) \cong \{0\}$. $\square$

2. **Problem 7.** Prove that the fundamental group of the real projective plane $\mathbb{R}P^2$ is isomorphic to $\mathbb{Z}_2$, a cyclic group of order 2.
   Likewise for real projective n-space, $\mathbb{R}P^n$.

   Solution:
   By the solution to Problem 5 above, we know that S^2 is simply connected. Also we have a two-sheeted covering map $p : S^2 \to \mathbb{R}P^2$ sending a pair of antipodal points on the sphere to their equivalence class in $\mathbb{R}P^2$. By Problem 11 of HW#2, this implies that for any $x \in \mathbb{R}P^2$ there's a set bijection $\pi_1(\mathbb{R}P^2) \to p^{-1}(x)$. Therefore $\pi_1(\mathbb{R}P^2)$ has two elements, and hence must be the group $\mathbb{Z}_2$, the only group of order two.

   As noted in the solution to Problem 5 above, $\pi_1(S^n) \cong \{0\}$ for all $n \geq 2$, so an analogous argument to the one in the previous paragraph gives that $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ for $n \geq 2$. For the lower dimensional cases : $\mathbb{R}P^1 \cong S^1$, so $\pi_1(\mathbb{R}P^1) \cong \mathbb{Z}$, and $\mathbb{R}P^0 \cong \{point\}$, so $\pi_1(\mathbb{R}P^0) \cong \{0\}$. $\square$

3. **Problem 14.** Prove that if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all $x \in S^2$, then g is onto.

Solution:

Suppose g is not onto, and let $q \in S^2$ be a point not in the image of g. Let $\phi : S^2 - \{q\} \to \mathbb{R}^2$ be a homeomorphism. Then $\phi \circ g : S^2 \to \mathbb{R}^2$ is a continuous map, so applying the Borsuk-Ulam theorem to this map gives that there's an $x \in S^2$ such that $\phi \circ g(x) = \phi \circ g(-x)$. Since ϕ is a homeomorphism and $image(\phi) \subset S^2 - \{q\}$, this implies that $g(x) = g(-x)$ for this point $x \in S^2$, a contradiction. $\square$

4. **Problem 18.** Show that, given a nonvanishing vector field on D^2, there exists a point of S^1 where the vector field points directly inward, and a point of S^1 where it points directly outward.

   Solution:

   Think of the vector field as a function $V : D^2 \to \mathbb{R}^2$. Since it is non-vanishing, we can define a map $F : D^2 \to S^1$, $F(x) = \frac{V(x)}{|V(x)|}$. Therefore the map $F|_{S^1} : S^1 \to S^1$ is null-homotopic, since it extends to the map F defined on the entire disc D^2 (which is a contractible space).

   Now, assuming that there's no point of S^1 where V points directly inward, we get that there's no point of S^1 for which $F(x) = -x$. Then $H(x,t) = \frac{tx+(1-t)F(x)}{|tx+(1-t)F(x)|}$ is well-defined (denominator is never zero), and is a homotopy between F and the identity map of S^1. But this is a contradiction, since the identity map induces an isomorphism on $\mathbb{Z} \cong \pi_1(S^1)$ while the constant map (and hence any null-homotopic map) induces the trivial homomorphism $1 \mapsto 0$.

   To show that there's a point of S^1 where V points directly outward, apply the above argument to the non-vanishing vector field $-V$, so we get that there's a point where $-V$ points directly inward! That's the same as saying that at this point, V points directly outward. $\square$