Math 601 Homework 7 Solutions to selected problems

1. **Problem 8.** If $i : A \to X$ is the inclusion of a retract of X, show that $i_* : H_k(A) \to H_k(X)$ is a monomorphism onto a direct summand of $H_k(X)$. If A is a deformation retract of X, show that i_* is an isomorphism.
   Solution:
   If $r : X \to A$ is a retraction, then we have : $r \circ i = id_A$ as a map from A to A. On homology, this is $r_* \circ i_* = (id_A)_* = id_{H_k(A)}$. Since $id_{H_k(A)}$ is injective, this implies that i_* must be injective as well. Therefore i_* is a monomorphism onto a subgroup $H = i_*(H_k(A))$ of $H_k(X)$. We still need to show that H is a direct summand of $H_k(X)$.

   Define the subgroup $K = ker(r_*)$ of $H_k(X)$. Then $H \cap K = \phi$. To see this, let $y \in H \cap K$. Since $y \in H$, $y = i_*(z)$ for some $z \in H_k(A)$. But since $y \in K$, $r_*(y) = 0$, so $r_* \circ i_*(z) = 0$, so $(id_A)_*(z) = 0$, so $z = 0$, which means y must be 0.

   Next, observe that any $x \in H_k(X)$ can be written as $x = i_* \circ r_*(x) + (x - i_* \circ r_*(x))$. The first term clearly lies in H. Also, $r_*(x - i_* \circ r_*(x)) = r_*x - r_*i_* \circ r_*(x) = r_*x - (id_A)_* \circ r_*(x) = 0$, so the second term lies in K.

   This shows that $H_k(X) = H \oplus K$.

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   If r is a deformation retraction then $i \circ r$ is homotopic to id_X, so on homology, $i_* \circ r_* = (id_X)_*$, which implies that i_* is surjective. Along with the previous part, this shows that i_* is an isomorphism. $\square$

2. **Problem 9.** Show that it is impossible to retract the ball B^n onto its boundary $\partial B^n = S^{n-1}$.
   Solution:
   Suppose there exists a retraction map $r : B^n \to S^{n-1}$. Then by the previous question, $i_* : H_{n-1}(S^{n-1}) \to H_{n-1}(B^n)$ is a monomorphism. But $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, and $H_{n-1}(B^n) \cong 0$ for all $n - 1 \geq 1$. Therefore we've got a monomorphism from $\mathbb{Z}$ to the trivial group, a contradiction.

   For the case of B^1 (a closed interval) and S^0 (disjoint union of two points), we see that any continuous function $r : B^1 \to S^0$ must map the entire interval onto one of the points (continuous image of a connected set must be connected) Hence r cannot be the identity map when restricted to the boundary S^0. $\square$

3. **Problem 10.** Show that every continuous map of B^n to itself must have a fixed point.
   Solution:
   Suppose $f : B^n \to B^n$ is a continuous map without fixed points. Then we get a

well-defined map $F : B^n \to \partial B^n$ as follows. For any $x \in B^n$, consider the ray starting at $f(x)$ and passing through x, and let $F(x)$ be the point where this ray hits the boundary. Then F is continuous, and by its construction, F is the identity map on $\partial B^n = S^{n-1}$. Therefore $F : B^n \to \partial B^n$ is a retraction map. But by the previous problem, there does not exist a retraction $F : B^n \to \partial B^n$. A contradiction. $\square$

4. **Problem 13.** Show that a continuous vector field on S^{2n} must vanish somewhere.
   Solution:
   Suppose $\mathbf{V}$ is a non-vanishing vector field on S^{2n}. Then for each $x \in S^{2n}$, $\frac{\mathbf{V(x)}}{|\mathbf{V(x)}|}$ is well-defined, and by translating the starting point of the vector to the origin, it can be thought of as a moving point on the sphere S^{2n} Also note that for each x, the vector $\mathbf{x}$ is orthogonal to the vector $\frac{\mathbf{V(x)}}{|\mathbf{V(x)}|}$. Therefore, if we define

   $$H(x,t) = cos(t)\mathbf{x} + sin(t)\frac{\mathbf{V(x)}}{|\mathbf{V(x)}|}$$

   then for each $t \in [0,\pi]$, $H(x,t)$ lies on the unit sphere S^{2n}. Also, $H(x,0) = x = id_{S^{2n}}(x)$, and $H(x,\pi) = -x = \alpha(x)$, where α is the antipodal map of the sphere. So H is a homotopy between $id_{S^{2n}}$ and α. But $\deg(id_{S^{2n}}) = 1$, and by the solution of **Problem 12.**, $\deg(\alpha) = (-1)^{2n+1} = -1$, so $id_{S^{2n}}$ and α cannot be homotopic. A contradiction.

   $\square$

5. **Problem 15. (Borsuk-Ulam theorem for S^n.)** Show there is no continuous antipode-preserving map $f : S^n \to S^{n-1}$.
   Solution:
   Suppose $f : S^n \to S^{n-1}$ is a continuous antipode-preserving map. Then the restriction of f to the equator $S^{n-1} \to S^{n-1}$ defines a antipode-preserving map of S^{n-1} to itself. Therefore by **Problem 14.**, $g = f\big|S^{n-1}$ cannot be homotoped to a constant map $S^{n-1} \to S^{n-1}$. In other words, $g_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$, which is actually $g_* : \mathbb{Z} \to \mathbb{Z}$, cannot be the zero homomorphism.

   On the other hand, $g = f \circ i$, so $g_* = f_* \circ i_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^n) \to H_{n-1}(S^{n-1})$, where i is the inclusion of the equator S^{n-1} into S^n. Therefore g_* factors through the group $H_{n-1}(S^n) = \{0\}$, which means g_* must be the zero homomorphism. A contradiction. $\square$