Math 601 Homework 7 Solutions to selected problems

1. **Problem (E)** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism which preserves distances between points, that is, $d(F(x), F(y)) = d(x, y)$ for all pair of points x and y in $\mathbb{R}^n$. Suppose, in addition, that $F(0) = 0$, where 0 denotes the origin in $\mathbb{R}^n$. Prove that F is an orthogonal linear transformation.
   Solution:
   Since $F(0) = 0$, we have for each $x \in \mathbb{R}^n$,
   $$|F(x)| = |F(x) - 0| = d(F(x), 0) = d(F(x), F(0)) = d(x, 0) = |x|$$
   Using this in $d(F(x), F(y)) = d(x, y)$ gives us :
   $$\begin{aligned} |F(x) - F(y)|^2 &= |x - y|^2 \\ \implies |F(x)|^2 - 2\langle F(x), F(y)\rangle + |F(y)|^2 &= |x|^2 - \langle x, y\rangle + |y|^2 \\ \implies \langle F(x), F(y)\rangle &= \langle x, y\rangle \end{aligned}$$
   Therefore F preserves the inner product between vectors in $\mathbb{R}^n$. Now we'll use this to show that F is linear. We first claim that for all $x, y \in \mathbb{R}^n$, $F(x+y) = F(x)+F(y)$, i.e. $|F(x + y) - F(x) - F(y)| = 0$. Using the fact that F preserves norm and inner product, we have :
   $$\begin{aligned} &|F(x + y) - F(x) - F(y)|^2 \\ =&|F(x + y)|^2 + |F(x)|^2 + |F(y)|^2 - 2\langle F(x + y), F(x)\rangle - 2\langle F(x + y), F(y)\rangle \\ &+ 2\langle F(x), F(y)\rangle \\ =&|x + y|^2 + |x|^2 + |y|^2 - 2\langle x + y, x\rangle - 2\langle x + y, y\rangle + 2\langle x, y\rangle \\ =&|(x + y) - x - y|^2 \\ =&0 \end{aligned}$$
   Therefore F is additive. That F respects scalar multiplication can be proved similarly by examining $|F(\lambda x) - \lambda F(x)|$. Therefore F is a linear map. We showed earlier that F preserves inner products; this implies that F is an orthogonal linear transformation. □

2. **Problem (H)** Use the Mayer-Vietoris sequence to compute the homology of real projective n-space $\mathbb{R}P^n$.
   Solution:
   We will use induction on the dimension n to show that

$$H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } i = n \text{ is odd} \\ \mathbb{Z}/2 & \text{if } 0 < i < n \text{ is odd} \\ \{0\} & \text{if } 0 < i < n \text{ is even} \\ \{0\} & \text{if } i = n \text{ is even} \end{cases}$$

Also for each n, homology of $\mathbb{R}P^n$ in dimensions greater than n is zero. Note that $H_0(\mathbb{R}P^n) = \mathbb{Z}$ by connectedness. We first give the proof for $n = 2$ and $n = 3$.

Homology of $\mathbb{R}P^2$: To show that $H_1(\mathbb{R}P^2) = \mathbb{Z}/2$ and $H_2(\mathbb{R}P^2) = \{0\}$ we use the Mayer-Vietoris (M-V) sequence. Cover $\mathbb{R}P^2$ by two open sets U and V defined as follows. $\mathbb{R}P^2$ can be thought of as an identification space obtained from a disk D^2 (the closed "northern hemisphere") by identifying points on the boundary S^1 via the antipodal map of S^1.

Define U to be the "northern hemisphere" minus the boundary, inside of $\mathbb{R}P^2$. This is an open ball, hence contractible, and its homologies after the zero-eth one, are all zero.

Define V to be the compliment of a small ball around the north pole, inside of $\mathbb{R}P^2$. This set is a tubular neighbourhood of the $\mathbb{R}P^1$ sitting inside $\mathbb{R}P^2$ as the space gotten by doing identifications on the boundary S^1 of the original disk. In particular, V deformation retracts onto $\mathbb{R}P^1$ (which is in fact homeomorphic to S^1), so for purposes of homology V can be treated as $\mathbb{R}P^1$ (i.e. S^1).

The intersection $U \cap V$ is homeomorphic to $S^1 \times I$, hence deformation retracts to S^1, and for purposes of homology can be treated as S^1. Notice however that this S^1 wraps **twice** around the circle $\mathbb{R}P^1$ described above.

We're now ready to use the M-V sequence with these sets. We have the long exact sequence (les):

$$H_2(U \cap V) \to H_2(U) \oplus H_2(V) \to H_2(U \cup V) \xrightarrow{\partial} H_1(U \cap V) \xrightarrow{\psi} H_1(U) \oplus H_1(V) \to \cdots$$

which is

$$H_2(S^1) \to H_2(D^2) \oplus H_2(\mathbb{R}P^1) \to H_2(\mathbb{R}P^2) \xrightarrow{\partial} H_1(S^1) \xrightarrow{\psi} H_1(D^2) \oplus H_1(\mathbb{R}P^1) \to \cdots$$

So we have

$$0 \to 0 \oplus 0 \to H_2(\mathbb{R}P^2) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\psi} 0 \oplus \mathbb{Z} \to \cdots$$

So we get that $H_2(\mathbb{R}P^2)$ injects into $\mathbb{Z}$ via ∂. Also, by the earlier observation that the S^1 generating $U \cap V$ wraps twice around the S^1 generating V, the map $\psi : \mathbb{Z} \to \mathbb{Z}$ is multiplication by 2, hence injective. Therefore $\ker(\psi) = 0$. By exactness, $\ker(\psi) = \text{im}(\partial)$, which implies that the image of ∂ is 0, so $H_2(\mathbb{R}P^2) = 0$.

For $H_1(\mathbb{R}P^2)$ we look at the following portion of the les :

$$H_1(U \cap V) \xrightarrow{\psi} H_1(U) \oplus H_1(V) \xrightarrow{\phi} H_1(\mathbb{R}P^2) \xrightarrow{\rho} H_0(U \cap V) \xrightarrow{\alpha} H_0(U) \oplus H_0(V)$$

which is

$$\mathbb{Z} \xrightarrow{\psi} 0 \oplus \mathbb{Z} \xrightarrow{\phi} H_1(\mathbb{R}P^2) \xrightarrow{\rho} \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z}$$

By the definition of the maps in the M-V sequence, α is the map $1 \mapsto (1, -1)$, which is injective. Therefore $\ker(\alpha) = 0$, so $\mathrm{im}(\rho) = 0$, i.e. ρ is the zero homomorphism. This implies that ϕ surjects onto $H_1(\mathbb{R}P^2)$. Therefore $H_1(\mathbb{R}P^2) = \mathbb{Z}/\ker(\phi) = \mathbb{Z}/\mathrm{im}(\psi) = \mathbb{Z}/2\mathbb{Z}$. Also, for dimensions greater than 2, the M-V sequence implies that the homology of $\mathbb{R}P^2$ is zero, since it is surrounded by zeros in the les. This completes the proof of the homology of $\mathbb{R}P^2$.

Homology of $\mathbb{R}P^3$: Analogously to the case of $\mathbb{R}P^2$, we consider $\mathbb{R}P^3$ as an identification space gotten from D^3 by antipodal identification on the boundary S^2. We take the open sets to be U = an open disk neighbourhood of the north pole, V = a neighbourhood of the $\mathbb{R}P^2$ inside of $\mathbb{R}P^3$ which is the quotient of the above-mentioned S^2. Then U has trivial homology in dimensions bigger than 0, V deformation retracts onto $\mathbb{R}P^2$, whose homology we computed above, and $U \cap V = S^2 \times I$ which deformation retracts onto S^2.

Now we have the M-V sequence :

$$H_3(U \cap V) \to H_3(U) \oplus H_3(V) \to H_3(\mathbb{R}P^3) \xrightarrow{\partial} H_2(U \cap V) \xrightarrow{\psi} H_2(U) \oplus H_2(V) \to \cdots$$

which is

$$0 \to 0 \oplus 0 \to H_3(\mathbb{R}P^3) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\psi} 0 \oplus 0 \to \cdots$$

Exactness implies that ∂ is an isomorphism, so $H_3(\mathbb{R}P^3) = \mathbb{Z}$. Next, we have :

$$H_2(U) \oplus H_2(V) \xrightarrow{\phi} H_2(\mathbb{R}P^3) \xrightarrow{\rho} H_1(U \cap V)$$

which is

$$0 \oplus 0 \xrightarrow{\phi} H_2(\mathbb{R}P^3) \xrightarrow{\rho} 0$$

which implies that $H_2(\mathbb{R}P^3) = 0$. Finally, we have :

$$H_1(U \cap V) \to H_1(U) \oplus H_1(V) \xrightarrow{f} H_1(\mathbb{R}P^3) \xrightarrow{g} H_0(U \cap V) \xrightarrow{h} H_0(U) \oplus H_0(V)$$

which is

$$0 \to 0 \oplus \mathbb{Z}/2 \xrightarrow{f} H_1(\mathbb{R}P^3) \xrightarrow{g} \mathbb{Z} \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z}$$

The map h is given by $1 \mapsto (1, -1)$, hence is injective. Therfore the image of g is zero, i.e. g is the zero homomorphism, which implies that f is surjective. Because of the 0 at the start of the sequence, f is also injective, hence an isomorphism. Hence $H_1(\mathbb{R}P^3) = \mathbb{Z}/2$. Note also that homologies in higher dimensions are zero by the same argument as for $\mathbb{R}P^2$.

Homology of $\mathbb{R}P^n$ Now that we have the base cases, we're ready to induct on the dimension. So, let's suppose the homologies of $\mathbb{R}P^k$, $k < n$, are as claimed. We need to prove for $\mathbb{R}P^n$. Define open sets U, V analogous to the earlier cases. Then for purposes of homology, $U = D^n$, $V = \mathbb{R}P^{n-1}$, and $U \cap V = S^{n-1}$.

The M-V sequence gives us :

$$H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(\mathbb{R}P^n) \to 0$$

where the last zero can be explained using the last sequence of the argument for $\mathbb{R}P^3$, where it's shown that g is the zero homomorphism. Also, we're assuming $n > 3$, so $H_1(U \cap V) = H_1(S^{n-1}) = 0$. So, we have :

$$0 \to 0 \oplus \mathbb{Z}/2 \to H_1(\mathbb{R}P^n) \to 0$$

which implies that $H_1(\mathbb{R}P^n) = \mathbb{Z}/2$. Next, for $1 < i < n-1$, we have:

$$H_i(U \cap V) \to H_i(U) \oplus H_i(V) \to H_i(\mathbb{R}P^n) \to H_{i-1}(U \cap V)$$

which is

$$0 \to 0 \oplus H_i(\mathbb{R}P^{n-1}) \to H_i(\mathbb{R}P^n) \to 0$$

since $H_i(S^{n-1}) = H_{i-1}(S^{n-1}) = 0$. Therfore, $H_i(\mathbb{R}P^n) \cong H_i(\mathbb{R}P^{n-1})$, and we get from the induction hypothesis that $H_i(\mathbb{R}P^n)$ is $\mathbb{Z}/2$ if i is odd, and it is 0 if i is even. Lastly, we have the sequence :

$$0 \to H_n(\mathbb{R}P^n) \to \mathbb{Z} \to H_{n-1}(U) \oplus H_{n-1}(V) \to H_{n-1}(\mathbb{R}P^n) \to 0$$

using the homology of $\mathbb{R}P^{n-1}$ and of S^{n-1}. If n is odd, then $n-1$ is even, so $H_{n-1}(U) \oplus H_{n-1}(V) = H_{n-1}(\mathbb{R}P^{n-1}) = 0$, which implies by exactness that $H_n(\mathbb{R}P^n) = \mathbb{Z}$ and $H_{n-1}(\mathbb{R}P^n) = 0$. If n is even, $n-1$ is odd, so $H_{n-1}(\mathbb{R}P^{n-1}) = \mathbb{Z}$, so the sequence looks like :

$$0 \to H_n(\mathbb{R}P^n) \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} 0 \oplus \mathbb{Z} \xrightarrow{\gamma} H_{n-1}(\mathbb{R}P^n) \to 0$$

Because the S^{n-1} generating $U \cap V$ wraps twice around the $\mathbb{R}P^{n-1}$ generating V, the map β is $1 \mapsto (0, -2)$, which is injective. Exactness then implies that α is the zero homomorphism, so $H_n(\mathbb{R}P^n) = 0$. Also, γ is surjective, so $H_{n-1}(\mathbb{R}P^n) = \mathbb{Z}/\ker(\gamma) = \mathbb{Z}/\mathrm{im}(\beta) = \mathbb{Z}/2\mathbb{Z}$. Note also that homologies in higher dimensions are zero by the same argument as for $\mathbb{R}P^2$. This completes the proof of the homology of $\mathbb{R}P^n$. □