

**MAT 114, Spring 2006 - First Midterm
Solutions**

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1. Consider the points A, B, C of coordinates $(1, 2, 1)$, $(-1, 0, 1)$ and $(2, 1, 0)$ respectively. The area of the triangle ABC is:

A. $\sqrt{3}$ B. $\sqrt{6}$ C. $2\sqrt{6}$ D. $2\sqrt{3}$ E. $\sqrt{15}$

Solution. B. The area of the triangle is:

$$\text{Area}(ABC) = |\overline{AB} \times \overline{AC}|/2.$$

Since $\overline{AB} \times \overline{AC} = \langle -2, -2, 0 \rangle \times \langle 1, -1, -1 \rangle = \langle 2, -2, 4 \rangle$, it follows $\text{Area}(ABC) = \sqrt{24}/2 = \sqrt{6}$.

2. Consider the points A, B, C of coordinates $(1, 2, 1)$, $(-1, 0, 1)$ and $(2, 1, 0)$ respectively. The coordinates of a point D such that $ABCD$ is a parallelogram with diagonal BD are:

A. $(4, 0, 3)$ B. $(4, 3, 0)$ C. $(3, 1, 4)$ D. $(1, 3, 4)$ E. $(1, 3, 0)$

Solution. B. If the coordinates of D are (x, y, z) , the fact that midpoints of the two diagonals AC and BD in the parallelogram coincide implies:

$$\frac{\langle 1, 2, 1 \rangle + \langle 2, 1, 0 \rangle}{2} = \frac{\langle -1, 0, 1 \rangle + \langle x, y, z \rangle}{2}$$

so $x = 4, y = 3, z = 0$.

3. The angle between the diagonal of a cube and one of the diagonals of its faces, starting from the same vertex, is:

A. $\pi/3$ B. $\pi/6$ C. $\pi/4$ D. $\cos^{-1} \frac{\sqrt{6}}{3}$ E. $\cos^{-1} \frac{\sqrt{3}}{3}$

Solution. D. Take the cube to be the one formed by the unit vectors i, j, k . Then a diagonal of the cube is $i + j + k$ and the diagonal of a face starting in the same vertex is $i + j$. From the properties of the dot product:

$$(i + j + k) \cdot (i + j) = 2 = \sqrt{3}\sqrt{2} \cos \theta,$$

hence $\cos \theta = 2/\sqrt{6} = \sqrt{6}/3$.

4. The equation of the plane passing through the origin and parallel to the lines

$$L_1 : \langle t, 1 - t, 1 + t \rangle \quad \text{and} \quad L_2 : \langle 2, 2t + 1, -t \rangle \quad \text{is:}$$

A. $x - y - 2z = 0$ B. $x + y - 2z = 0$ C. $x - 2y + 2z = 0$ D. $x + 2y - 2z = 0$ E. $x + 2y + 2z = 0$

Solution. A. The normal of the plane is perpendicular to the direction vectors of the two lines, which are obtained by looking at the coordinates of t : $\langle 1, -1, 1 \rangle$ and $\langle 0, 2, -1 \rangle$. Therefore the normal vector is:

$$\mathbf{n} = \langle 1, -1, 1 \rangle \times \langle 0, 2, -1 \rangle = \langle -1, 1, 2 \rangle$$

and since the plane passes through $(0, 0, 0)$, its equation is: $-x + y + 2z = 0$.

5. In cylindrical coordinates, the equation $r = 2$ describes:

A. A plane B. A paraboloid C. A cone D. A sphere E. A cylinder

Solution. E. In rectangular coordinates, $r = \sqrt{x^2 + y^2}$, so $x^2 + y^2 = 4$ which is a cylinder centered around the z -axis.

6. In spherical coordinates, the equation $\rho = \cos \varphi$ describes (transform the equation in rectangular coordinates before answering):

A. A plane B. A paraboloid C. A cone D. A sphere E. A cylinder

Solution. D. The equation can also be written $\rho^2 = \rho \cos \varphi$, which in rectangular coordinates becomes:

$$x^2 + y^2 + z^2 = z, \quad \text{or} \quad x^2 + y^2 + (z - 1/2)^2 = 1/4,$$

which is a sphere centered at $(0, 0, 1/2)$ of radius $1/2$.

7. The two curves

$$C_1 : x = t, y = t^2, z = t^3 \text{ and } C_2 : x = \sin t, y = -1 + \cos 2t, z = t$$

intersect at the origin. The angle between their tangent vectors at the origin is:

- A. $\pi/3$ B. $\pi/6$ C. $\pi/4$ D. $\cos^{-1} \frac{\sqrt{6}}{3}$ E. $\cos^{-1} \frac{\sqrt{3}}{3}$

Solution. C. The tangent vectors to C_1, C_2 are $v_1(t) = \langle 1, 2t, 3t^2 \rangle$, $v_2(t) = \langle \cos t, -2 \sin 2t, 1 \rangle$. At $t = 0$, $v_1 = \langle 1, 0, 0 \rangle$, $v_2 = \langle 1, 0, 1 \rangle$, and the angle in between is:

$$\cos(\theta) = \frac{v_1 \cdot v_2}{|v_1||v_2|} = 1/\sqrt{2}.$$

8. A spaceship starts moving from the origin at $t = 0$ with an initial velocity $\mathbf{v}(0) = \langle 1, 0, 1 \rangle$. The acceleration of the spaceship depends on time as follows: $\mathbf{a}(t) = \langle 0, 1, t \rangle$. What are the coordinates of the spaceship at time $t = 1$?

- A. $(1, 1/2, 7/6)$ B. $(1, 1, 3/2)$ C. $(1, 2, 3/2)$ D. $(2, 1, 1/2)$ E. $(0, 1, 1/6)$

Solution. A.

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \langle C_1, C_2 + t, C_3 + t^2/2 \rangle$$

Since $\mathbf{v}(0) = \langle 1, 0, 1 \rangle$, we get $\mathbf{v}(t) = \langle 1, t, 1 + t^2/2 \rangle$.

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \langle t + C_1, t^2/2 + C_2, t + t^3/6 + C_3 \rangle$$

and since $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ (the spaceship starts moving from the origin) we have: $\mathbf{r}(t) = \langle t, t^2/2, t + t^3/6 \rangle$. Hence $\mathbf{r}(1) = \langle 1, 1/2, 7/6 \rangle$.

9. Sketch and describe in words the following surfaces:

- (a) $x^2 + y^2 + z^2 = 2x + 2y + 2z$
 (b) $x^2 - 2y^2 - 4z^2 = 0$
 (c) $x^2 - 2y^2 - 4z^2 = 5$
 (d) $x - 2y^2 - 4z^2 = 1$

Solution. (a) Completing the square: $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 3$, which is a sphere of radius $\sqrt{3}$, centered at $(1, 1, 1)$.

(b) An elliptic cone with opening around the x -axis. The sections in planes $x = \text{constant}$, are ellipses, while for $z = 0$, $x = \pm\sqrt{2}y$ are two lines.

(c) A hyperboloid of two sheets, around the x -axis. The sections in planes $x = \text{constant}$, are ellipses, if $x^2 - 5 = 2y^2 + 4z^2 \leq 0$, that is $|x| \leq \sqrt{5}$. If $z = 0$, $x^2 - 2y^2 = 5$ is a hyperbola opening along the x -axis.

(d) Paraboloid, opening in the positive x direction, with the tip at $(1, 0, 0)$.

10. Is there a point on the curve

$$\mathbf{r}(t) = \langle 2t, t^2, t^3 \rangle$$

where the tangent line to the curve is perpendicular to the plane

$$\frac{x}{2} + \frac{y}{2} + \frac{z}{3} = 1?$$

If your answer is yes, find its coordinates. If it is no, justify your answer.

Solution. The tangent vector to the curve is $\mathbf{v} = \langle 2, 2t, 3t^2 \rangle$ and the normal vector to the plane is $\mathbf{n} = \langle 1/2, 1/2, 1/3 \rangle$. The tangent vector is normal to the plane when it is parallel to the normal vector, that is when:

$$\langle 2, 2t, 3t^2 \rangle = c \langle 1/2, 1/2, 1/3 \rangle,$$

where c is a constant. Comparing the x coordinates we find $c = 4$, and the y coordinates give $t = 1$. But then the z coordinates are $3 \neq 4/3$, so there is no point on the curve when the two vectors are parallel.

Warning! There were two common mistakes. One was to assume that the tangent vector is normal to the plane when $\mathbf{v} \cdot \mathbf{n} = 0$, which is the condition for tangent vector to be parallel to the plane.

The other common wrong assumption is that two vectors are parallel if they equal, so many people concluded that since $\mathbf{v} \neq \mathbf{n}$, the two vectors are not parallel. Instead, two vectors are parallel if one is a scalar multiple of the other, as shown above.