MAT 114, Spring 2006 - Second Midterm Solutions
Thursday, March 1st
Instructor: Alexandru Popa

---

1. a) (7 pts) Find the point on the surface $z = x^2 + 2y^2$ in the first octant, which is closest to the point $(0, 0, 2)$.

   A. $(0, 0, 0)$ B. No closest point exists. C. $(0, \sqrt{7/8}, 7/4)$
   D. $(\sqrt{3/2}, 0, 3/2)$ E. $(1, 1/\sqrt{2}, 2)$ F. $(1, 1/2, 3/2)$

   b) (3 pts) Find the point on the surface $z = x^2 + 2y^2$ in the first octant which is farthest from the point $(0, 0, 2)$.

   A. $(0, 0, 0)$ B. No farthest point exists. C. $(0, \sqrt{7/8}, 7/4)$
   D. $(\sqrt{3/2}, 0, 3/2)$ E. $(1, 1/\sqrt{2}, 2)$ F. $(1, 1/2, 3/2)$

   Answer. a) C, b) B.

   The problem asks for the extrema of the function

$$f(x, y, z) = x^2 + y^2 + (z - 2)^2, \text{subjection to the constraint } \ g(x, y, z) = x^2 + 2y^2 - z = 0.$$

   Since the paraboloid $z = x^2 + 2y^2$ is unbounded (there are points on it with any large values of x and y), there is no farthest point. By the method of Lagrange multipliers, the closest point is among the solutions to the system:

$$\nabla f = \lambda \nabla g, z = x^2 + 2y^2.$$

   We get 4 equations:

$$2x = 2\lambda x, \quad 2y = 4\lambda y, \quad 2(z - 2) = -\lambda, \quad x^2 + 2y^2 = z.$$

   The first equation implies that either $x = 0$, or $\lambda = 1$.

   Case 1. $x = 0$: The second equation implies either $y = 0$ or $\lambda = 1/2$. Case 1a. $x = 0$, $y = 0$. Then $z = 0$ as well, so we get the solution $(0, 0, 0)$. Case 1b. $x = 0$, $\lambda = 1/2$. Then $z = 7/4$ from the 3rd equation, and $y = \pm\sqrt{7/8}$ from the 4th equation. We get the solution $(0, \pm\sqrt{7/8}, 7/4)$.

   Case 2. $\lambda = 1$. The 2nd equation implies $y = 0$, the 3rd equation implies $z = 3/2$, and the 4th implies $x = \pm\sqrt{3/2}$. We get the solution $(\pm\sqrt{3/2}, 0, 3/2)$.

   We found five solutions of the Lagrange multiplier system, only three of which are in the first octant. The minimum of f among these points occurs at $(0, \sqrt{7/8}, 7/4)$, when $f = 15/16$.

2. The surfaces $xyz^2 = 4$ and $3x^2 + y^2 = z^2$ intersect along a curve C. A tangent vector to the curve C at the point $(1, 1, 2)$ is:

A. $< 2, 3, -1 >$ B. $< 3, -5, 2 >$ C. $< 3, 5, 2 >$
D. $< 2, -3, 1 >$ E. $< 1, 4, 2 >$ F. $< 3, 5, 4 >$

Answer. B

Since the curve of intersection lies on both surfaces, the normal vectors to both surfaces are perpendicular to the curve, that is to its the tangent vector $\mathbf{v}$. If $f = xyz^2$, $g = 3x^2 + y^2 - z^2$, the normal vectors to the two surfaces are

$$\nabla f =< yz^2, xz^2, 2xyz >, \quad \nabla g =< 6x, 2y, -2z >$$

At the point $(1, 1, 2)$, these vectors are $< 4, 4, 4 >$ and $< 6, 2, -4 >$. A tangent vector to the curve of intersection is perpendicular to both these vectors, hence one such vector is the cross product :

$$\mathbf{v} =< 4, 4, 4 > \times < 6, 2, -4 >=< -24, 40, -16 >= -8 < 3, -5, 2 >$$

The answer is $< 3, -5, 2 >$.

3. The equation of the tangent plane to the surface $x^3y + y^3z + z^3x = -1$ at the point $(1, 1, -1)$ is

A. $x + y + 2z = 2$ B. $-x + y + 3z = 1$ C. $x + y - z = 3$
D. $x - y - 2z = 0$ E. $x - y + 2z = -2$ F. $x - y + z = -1$

Answer. E.

The surface is a level set of the function $f = x^3y + y^3z + z^3x$, hence the vector $\nabla f =< 3x^2y + z^3, x^3 + 3y^2z, y^3 + 3z^2x >$ is normal to the surface. A normal to the tangent plane at the point $(1, 1, -1)$ is

$$\nabla f(1, 1, -1) =< 2, -2, 4 > .$$

The equation of the tangent plane is:

$$2(x - 1) - 2(y - 1) + 4(z + 1) = 0$$

which simplifies to: $x - y + 2z = -2$.

4. Let $f(x, y) = e^x y + y^2 x$. Estimate $f(0.1, 0.9)$, using the linear approximation of f at the point $(0, 1)$.

A. 1.3
B. 1.1
C. 0.9
D. 1.2
E. 1
F. 0.8

Answer. B.

The linearization of f at $(0, 1)$ is:

$$L(x, y) = f(0, 1) + f_x(0, 1)x + f_y(0, 1)(y - 1) = 1 + 2x + y - 1$$

Plugging in $x = 0.1$, $y = 0.9$, we get the answer, 1.1.

5.

$$\lim_{(x,y)\to(0,0)} x^2 \ln(1 + x^2 + y^2) =$$

A. $\ln 2$ B. 1 C. -1
D. 0 E. Does not exist F. 2

Answer. D.

Since the function inside the limit is continuous at $(0, 0)$, we find the limit to be 0 by plugging in $x = 0$, $y = 0$.

6. Consider the function $f(x, y) = x^2 + 2y^2 - 2y$, defined on the region D bounded by the ellipse $\frac{x^2}{4} + y^2 = 1$.

a) (5pts) The absolute maximum of f on D is:

A. 5.5	B. 4.5	C. 3.5
D. 4	E. It does not exist.	F. 5.5

b) (5pts) The absolute minimum of f on D is:

A. -1	B. -1.5	C. -0.5
D. 0	E. It does not exist.	F. -2.5

Answer. a) B., b) C.

The critical points inside the region are where $f_x = 2x = 0$, $f_y = 4y - 2 = 0$. We get one point, $(0, 1/2)$, and

$$f(0, 1/2) = -1/2.$$

To find the critical points on the boundary $\frac{x^2}{4} + y^2 = 1$, we substitute $x^2 = 4(1 - y^2)$ in the formula for f, obtaining the function:

$$g(y) = 4(1 - y^2) + 2y^2 - 2y = -2y^2 - 2y + 4, \text{ where } -1 \leq y \leq 1.$$

The critical points of this function are $y = \pm 1$, and where $g'(y) = 0$, that is $y = -1/2$. The corresponding values of x can be found from the equation $x^2 = 4(1 - y^2)$, and we get the points: $(0, \pm 1), (\pm\sqrt{3}, -1/2)$. The values of f at these points are

$$f(0, 1) = 0, f(0, -1) = 4, f(\pm\sqrt{3}, -1/2) = 4.5$$

Comparing the values of f at these points and at the critical point inside, we find the absolute maximum to be 4.5 and the absolute minimum to be -0.5.

7. The volume of the solid in the first octant, bounded by the planes $z = 0$, $y = 0$, $y = x$, and by the cylinder $x^2 + z^2 = 1$, is:

A. 2/3	B. 1/3	C. 3/2
D. 1	E. 1/2	F. 1/4

Answer. B

The volume equals

$$\int_0^1 \int_0^x \sqrt{1-x^2}\,dy\,dx = \int_0^1 x\sqrt{1-x^2}\,dx$$

Substituting $u = 1 - x^2$ in the last integral we get:

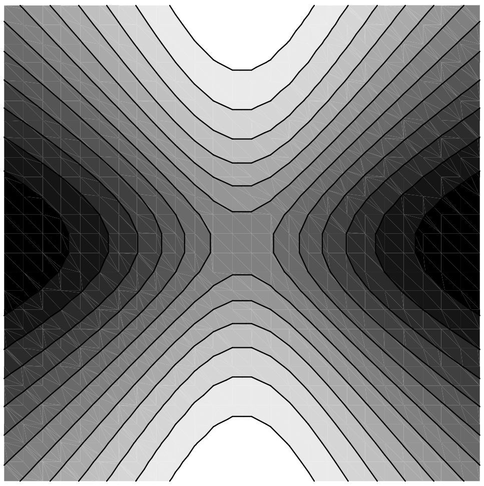
$$\int_0^1 \sqrt{u}\,du/2 = 1/3$$

8. Match the following functions with the level set graphs below. Darker shades represent level sets where the functions take higher values. In each graph, the origin is at the center of the square.

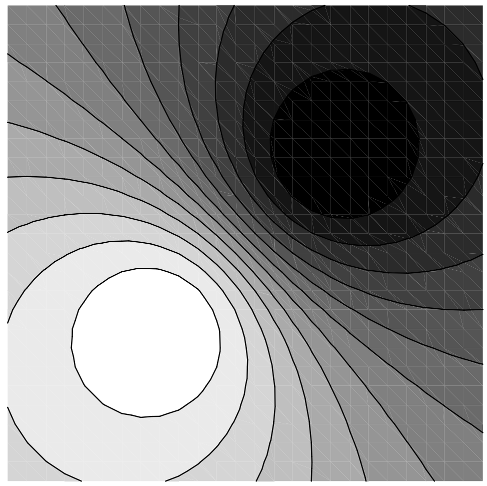
   Indicate your answer by writing the numbers 1-4 corresponding to the functions, in the space provided under each graph (you do not need to record your answer on the front page).

1. $f_1(x,y) = \dfrac{x^2 - y}{x^2+1}$ **2.** $f_2(x,y) = \dfrac{x^2 - y^2}{x^2+y^2+1}$

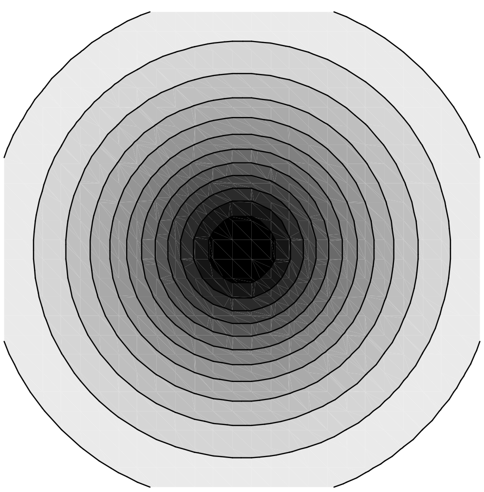
3. $f_3(x,y) = \dfrac{x + y}{x^2+y^2+1}$ **4.** $f_4(x,y) = \dfrac{1}{x^2+y^2+1}$



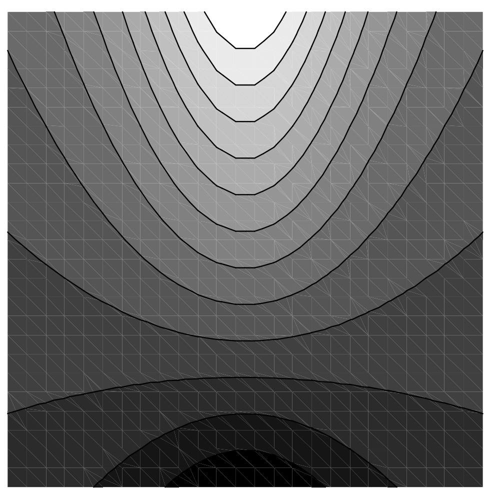
2



3



4



1

For the function f_1 the level sets have equations:

$$\frac{x^2 - y}{x^2+1} = C, \text{ that is } y = (1-C)x^2 - C,$$

where C is a constant. The level sets are parabolas (except for $C = 1$, when the level set is $y = -1$), therefore the corresponding graph is the last one. Similarly one sees that f_2 has hyperbolas for levels sets, f_3 has circles centered on the line $y = x$, and f_4 has circles centered at the origin.

9. The temperature in space is given by the function $T(x, y, z) = x^2y + z$. A particle moves on a trajectory $\mathbf{r}(t) =< x(t), y(t), z(t) >$, and at time $t = 0$ its position and velocity are $\mathbf{r}(0) =< 1, 2, 10 >$ and $\mathbf{v}(0) =< 1, 2, -1 >$ respectively. Viewing the temperature T along the trajectory as a function of t, what is $dT/dt(0)$?

A. 0	B. 5	C. 3
D. -2	E. 2	F. -5

Answer: B.

We use the chain rule:

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{dz}\frac{dz}{dt}$$

We compute

$$\frac{\partial T}{\partial x}(1, 2, 10) = 4, \frac{\partial T}{\partial y}(1, 2, 10) = 1, \frac{\partial T}{\partial y}(1, 2, 10) = 1$$

and is is given that at $t = 0$, $< \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} >=< 1, 2, -1 >$. Hence $dT/dt(0) = 5$.

10. Let D be the region in the first quadrant bounded by the curves $x = 0$, $y = 9$, and $y = x^2$. The value of the integral

$$\iint_D 4x\cos(y^2)dA \quad \text{is:}$$

A. $1/2$

B. $\sin 9$

C. $\sin 81$

D. $\cos 81$

E. $\cos 9$

F. $\sqrt{3/2}$

Answer: C.

In order to be able to compute the integral, we need to set it up with respect to $dxdy$, since $\cos y^2$ has no antiderivative in terms of simple functions. Therefore:

$$\iint_D 4x\cos(y^2)dA = \int_0^9 \int_0^{\sqrt{y}} 4x\cos(y^2)dxdy = \int_0^9 2y\cos(y^2)dy = \sin 81.$$