1. *(15 points)* Let R be the solid region above the cone $z = \sqrt{x^2+y^2}$ and below the plane $z = 1$.

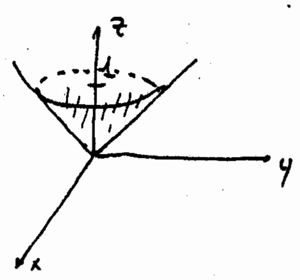
(a) Give a rough sketch of R.

(b) Describe R using cylindrical coordinates.

(c) Find the volume of R.

(d) Find the average value of z on R (the average of a function f on R is $\int_R f dV/\text{Volume}(R)$).

(a)



(b) Cylindrical coord: $z = z$, $x = r\cos\theta$, $y = r\sin\theta$.

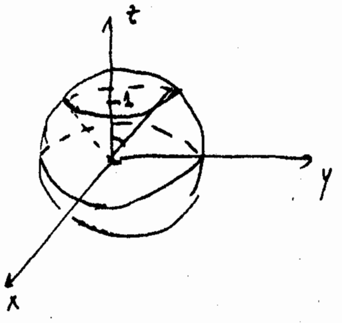
$R: \{(r\cos\theta, r\sin\theta, z) : 0 \le r \le z,\ 0 \le \theta < 2\pi,\ 0 \le z \le 1\}$.

(c) Volume in cylindrical coord: $\int_0^1 \int_0^{2\pi} \int_0^z 1 \cdot r\,dr\,d\theta\,dz = \int_0^1 \int_0^{2\pi} \frac{z^2}{2}\,d\theta\,dz = \frac{\pi}{3}$.

(d) $\iiint_R z\,dV = \int_0^1 \int_0^{2\pi} \int_0^z z\,r\,dr\,d\theta\,dz = \int_0^1 \int_0^{2\pi} \frac{z^3}{2}\,d\theta\,dz = \frac{\pi}{4}$

So $\bar{z} = \frac{\pi/4}{\pi/3} = \frac{3}{4}$

2. *(10 points)* Using whatever method you like, find the volume of the solid region inside the sphere $x^2 + y^2 + z^2 = 4$ and below the plane $z = 1$.



Use spherical coord: $x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$.

To find the angle φ corresponding to the intersection of the sphere with the plane, solve for $\cos\theta = \frac{z}{\rho} = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$.

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{\frac{1}{\cos\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta + \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\pi}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

accounts for the cone part

Plane equation: $z = 1 = \rho\cos\phi$, so

$\rho = \frac{1}{\cos\phi}$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}\frac{\sin\phi}{(\cos\phi)^3} \, d\phi \, d\theta + \int_0^{2\pi}\int_{\pi/3}^{\pi} \frac{8}{3}\sin\phi \, d\phi \, d\theta$$

Use substitution $u = \cos\phi$ for the first integral.

$$= \pi + 8\pi$$

$$= 9\pi$$

3. *(15 points)* Let R be the square in the uv plane withvertices $(1,1)$, $(2,1)$, $(1,2)$, and $(2,2)$. Let F be the mapping from the uv plane to the xy plane given by:

$$\begin{aligned} x &= uv \\ y &= \frac{u}{v} \end{aligned}$$

(a) Show that the images of the lines $u = 1$ and $u = 2$ under F lie on the curves $xy = 1$ and $xy = 4$ respectively.

(b) Show that the images of the lines $v = 1$ and $v = 2$ under F lie on the curves $x = y$ and $x = \cancel{2}4y$ respectively.

(c) Let S be the image of R under F. Sketch S.

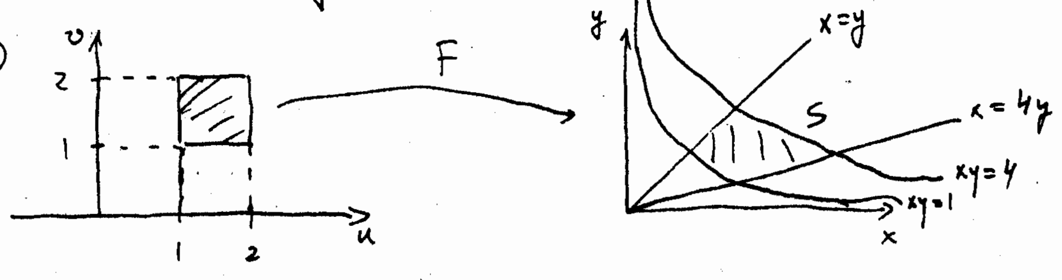
(d) Compute

$$\int_S \frac{x}{y}\, e^{xy}\, dA$$

(a), (b) Solving for u, v in terms of x, y we get: $u^2 = xy$, $v^2 = \frac{x}{y}$ (multiply and divide the given eqns).

when $u = 1$: $xy = 1$
$u = 2$: $xy = 4$
$v = 1$: $x = y$
$v = 2$: $x = 4y$

(c)



(d) Using the substitution $\begin{cases} x = uv \\ y = \frac{u}{v} \end{cases}$, the region S gets mapped to the given rectangle,

so $\displaystyle\int_S \frac{x}{y} e^{xy}\, dA = \int_1^2\int_1^2 v^2 e^{u^2} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv = \int_1^2\int_1^2 2uv e^{u^2} du\, dv =$

$\displaystyle = \int_1^2 v \cdot \left(e^{u^2}\Big|_1^2\right) dv =$

$= (e^4 - e)\cdot\frac{3}{2}$.

where $\frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian: $\begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -2\frac{u}{v}$